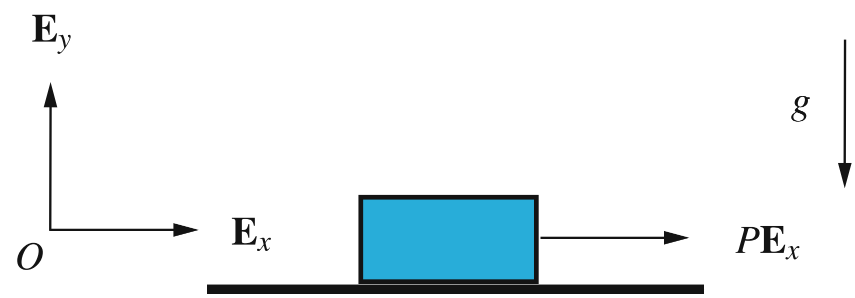
Chapter 4: Friction Forces and Spring Forces

Topics: - friction forces
- spring forces

4.1 An Experiment on Friction

A block of mass m on a surface with a force $P\overline{E}_x$.
What do we observe?



a) small $P \longrightarrow$ block remains at rest
b) beyond some $P = P^* \longrightarrow$ block starts to move
c) once moving $\longrightarrow P = P^{**}$ required to keep it going at a constant speed
d) P^* and P^{**} are proportional to the normal force $\overline{N}$.

Let's model these phenomena.
1) Kinematics

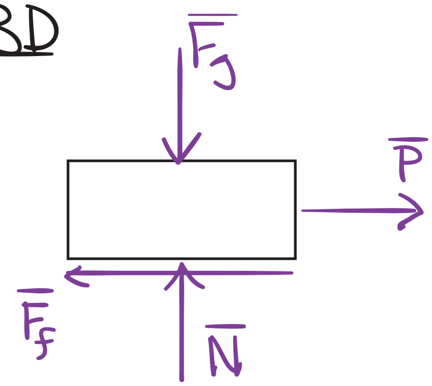
$$\overline{r} = x\overline{E}_x + y_0\overline{E}_y + z_0\overline{E}_z$$

$$\overline{v} = \dot{\overline{r}} = \dot{x}\overline{E}_x$$

$$\overline{a} = \dot{\overline{v}} = \ddot{x}\overline{E}_x$$

2) Forces

FBD



$\bar{F}_g, \bar{P}, \bar{F}_f, \bar{N}$

$F_g = -mg\bar{E}_y$

$\bar{P} = P\bar{E}_x$

$P > 0$

What do we know about $\bar{N}$ and $\bar{F}_f$?

$$\bar{N} = N\bar{E}_y$$

$\bar{F}_f$ is proportional to $\bar{N}$ and opposing motion in the x-direction:

$$\bar{F}_f = F_{fx}\bar{E}_x \qquad (F_{fx} < 0)$$

The two cases give different coefficients:

Static $(P < P^*)$: $\mu = \mu_s$

Dynamic $(P = P^{**})$: $\mu = \mu_d$

3) $\bar{F} = m\bar{a}$ in Cartesian coordinates

$$\begin{bmatrix} P + F_{fx} \\ N - mg \\ 0 \end{bmatrix} = m \begin{bmatrix} \ddot{x} \\ 0 \\ 0 \end{bmatrix}$$

4) Analysis

$N = mg$ $\qquad\qquad$ $F_{fx} = m\ddot{x} - P$

If the block is not moving,

$\ddot{x} = 0 \Rightarrow \overline{F_f} = -P\overline{E_x}$ while $P \leq P^* = \mu_s mg$.

If the block is moving at a constant speed,

$\ddot{x} = 0 \Rightarrow \overline{F_f} = -P\overline{E_x} = -\mu_d |N| E_x$

If the block is accelerating,

$\overline{F_f} = (m\ddot{x} - P)\overline{E_x}$.

The coefficients of static friction μ_s and dynamic friction μ_d depend on the nature of the surface and the block. They are determined experimentally.

4.2 Static and Dynamic Coulomb Friction Forces

The above applies only to flat, stationary surfaces.
The following theory applies to:

(a) cases where the surface is curved,
(b) cases where the particle is moving on a space curve,
(c) cases where the spacecurve or surface is moving.

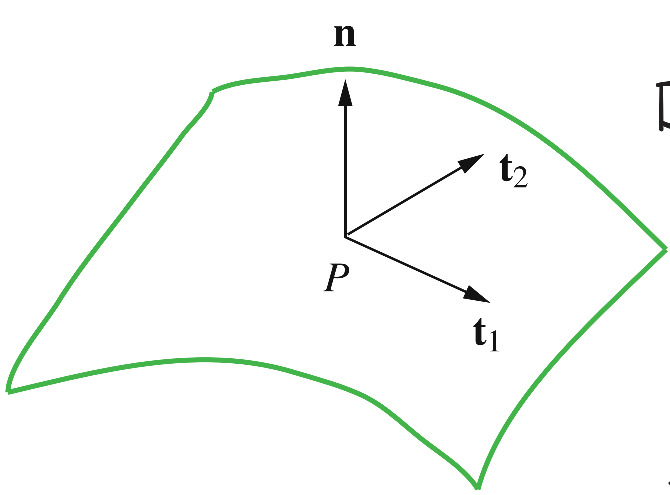
Notation

$\bar{r}, \bar{v}, \bar{a}$ — position, velocity, and acceleration of the particle

$\bar{v}_c$ — velocity of the space curve at the point-of-contact between the curve + particle

$\bar{v}_s$ — velocity of the surface at the point-of-contact between the surface + particle

4.2.1 Particle on a Surface



Define: $\bar{n}$ — normal basis vec.
$\bar{t}_1$ — tangential basis vec.
$\bar{t}_2$ — tangential basis vec.

How can we model friction for a particle on this surface?

Define: $\overline{V}_{rel} = \overline{V} - \overline{V}_s$.

If $\overline{V}_{rel} = \overline{0}$ $\longrightarrow$ static friction.
If $\overline{V}_{rel} \neq \overline{0}$ $\longrightarrow$ dynamic friction.

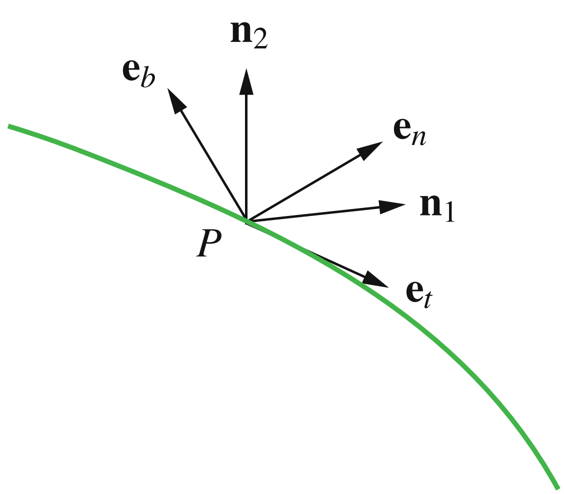
The amount of static friction force is limited by:

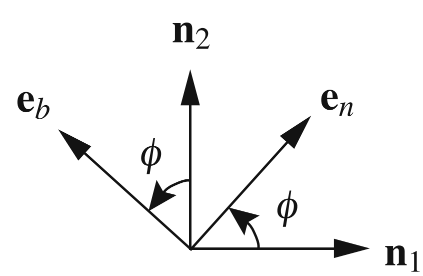
$$\boxed{\|\overline{F}_f\| \leq \mu_s \|\overline{N}\|} .$$

If this is false, $\overline{V}_{rel} \neq \overline{0}$ and

$$\boxed{\overline{F}_f = -\mu_d \|\overline{N}\| \frac{\overline{V}_{rel}}{\|\overline{V}_{rel}\|}} .$$

4.2.2 A Particle on a Space Curve





Define $\overline{n}_1, \overline{n}_2$ — normal basis vectors in the $\overline{e}_n - \overline{e}_b$ plane.

Define: $\overline{V}_{rel} = \overline{V} - \overline{V}_c$

If $\bar{v}_{rel} = \bar{0} \rightarrow$ static friction.
If $\bar{v}_{rel} \neq \bar{0} \rightarrow$ dynamic friction.

The amount of static friction force is limited by:

$$\|\bar{F}_f\| \leq \mu_s \|\bar{N}\| .$$

If this is false, $\bar{v}_{rel} \neq \bar{0}$ and

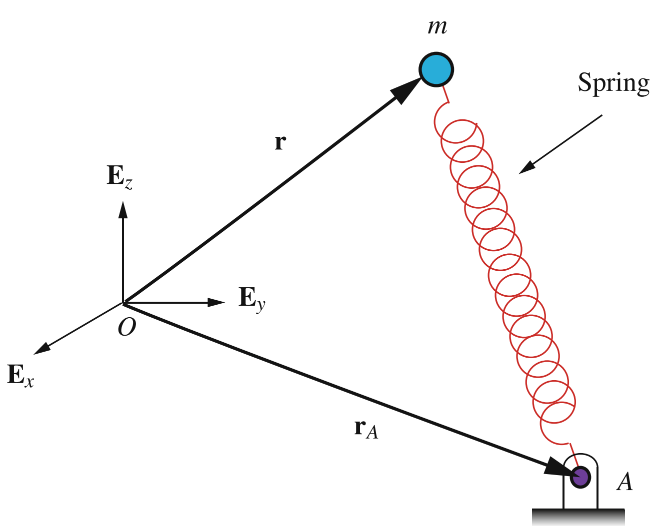
$$\bar{F}_f = -\mu_d \|\bar{N}\| \frac{\bar{v}_{rel}}{\|\bar{v}_{rel}\|} .$$

4.4 Hooke's Law and Linear Springs

Hooke's Law in modern terms:

The force from a spring is proportional to its extension. ("Ut tensio sic vis.")

We call the contant of proportionality K.



Let's explore the dynamics of springs in their "linear" regime.

Assume springs to be massless with unstretched length L.

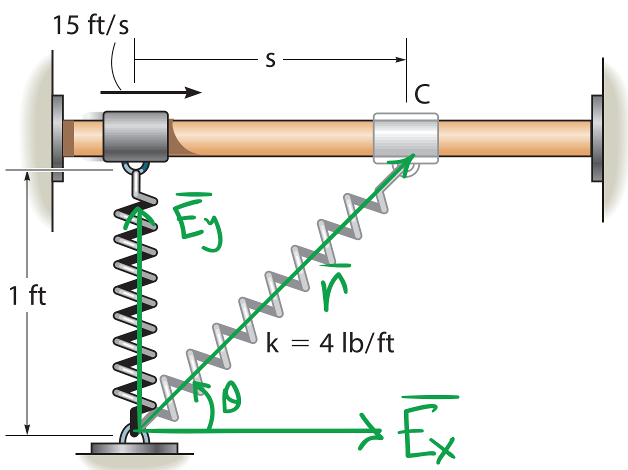
With position vectors as shown above, Hooke's Law can be written:

$$\|\overline{F}_s\| = |K(\|\overline{r} - \overline{r}_A\| - L)|.$$

The force vector is:

$$\overline{F}_s = -K(\|\overline{r} - \overline{r}_A\| - L)\,\frac{\overline{r} - \overline{r}_A}{\|\overline{r} - \overline{r}_A\|}.$$

13-36 (Hibbeler) Example



Given the 2lb collar on the smooth rod with the spring of unstretched length 1ft, find $\hat{v}(s)$, the velocity as a function of arclength if $\hat{v}(s=0) = v_0$. Also find $\hat{N}(s)$, the force from the rod.

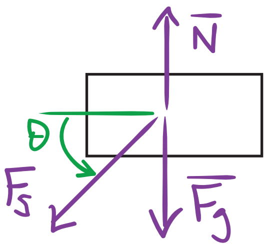
Kinematics

$$\overline{r} = x\overline{E}_x + y\overline{E}_y, \quad \overline{v} = \dot{x}\overline{E}_x + \dot{y}\overline{E}_y, \quad \overline{a} = \ddot{x}\overline{E}_x + \ddot{y}\overline{E}_y$$

$$\hat{r}(s) = s\overline{E}_x + L\overline{E}_y \qquad \|\hat{r}(s)\| = \sqrt{s^2 + L^2}$$

$$\hat{v}(s) = \dot{\hat{r}} = \dot{s}\overline{E}_x \qquad \hat{a}(s) = \dot{\hat{v}} = \ddot{s}E_x$$

Forces



$$\bar{F}_g = -mg\bar{E}_y$$

$$\bar{F}_s = -K\left(\|\bar{r}\| - L\right)\frac{\bar{r}}{\|\bar{r}\|} \quad \text{where } L = 1\text{ ft.}$$

$$= -K\frac{\left(\sqrt{s^2+L^2} - L\right)}{\sqrt{s^2+L^2}}\left(s\bar{E}_x + L\bar{E}_y\right)$$

$\bar{F} = m\bar{a}$

In the Cartesian basis,

$$\begin{bmatrix} -Ks\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2} \\ N - mg - KL\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2} \end{bmatrix} = m\begin{bmatrix} \ddot{s} \\ 0 \end{bmatrix}$$

Analysis

The $\bar{E}_x$ equation is the equation of motion:

$$-Ks\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2} = m\ddot{s} = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

Separate and solve:

$$-K\int_0^s \frac{\sigma\sqrt{\sigma^2+L^2} - L\sigma}{\sqrt{\sigma^2+L^2}}\,d\sigma = m\int_{v_0}^{v} u\,du$$

$$-K\left(\tfrac{1}{2}s^2 - L\sqrt{s^2+L^2}\right) - L^2K = m\left(v^2 - v_0^2\right)$$

$$\boxed{\hat{v}(s) = \pm\left(v_0^2 - \frac{K}{m}\left(\tfrac{1}{2}s^2 - L\sqrt{s^2+L^2}\right) - L^2K/m\right)^{1/2}}$$

The $\overline{E}_y$ equation gives

$$\boxed{\hat{N}(s) = \left(mg + KL\left(\sqrt{s^2+L^2} - L\right)/\sqrt{s^2+L^2}\right)\overline{E}_y} \; .$$