8.1.3 Angular Velocity + Acceleration Vectors

Recall: $(\bar{x}_1 - \bar{x}_2) = Q(\bar{X}_1 - \bar{X}_2)$. $\quad (*)$

Therefore: $(\bar{X}_1 - \bar{X}_2) = Q^T(\bar{x}_1 - \bar{x}_2)$. $\quad (**)$

Differentiating the equation $(*)$ with respect to time, we can the relative velocity of the two points:

where $\bar{v}_1 = \dot{\bar{x}}_1$ and $\bar{v}_2 = \dot{\bar{x}}_2$. Subsituting $(**)$ into this equation and recalling our expression for $\dot{Q}Q^T$ from section 8.1.2, we get:

$$(\bar{v}_1 - \bar{v}_2) = \dot{Q}Q^T(\bar{x}_1 - \bar{x}_2)$$

$$= \begin{bmatrix} 0 & -\Omega_{21} & \Omega_{13} \\ \Omega_{21} & 0 & -\Omega_{32} \\ -\Omega_{13} & \Omega_{32} & 0 \end{bmatrix} (\bar{x}_1 - \bar{x}_2).$$

where $\bar{\omega} = \Omega_{32}\bar{E}_x + \Omega_{13}\bar{E}_y + \Omega_{21}\bar{E}_z$ is called the angular velocity vector. $\bar{\omega}$ is the same for relating any two points in a body, and is a function of time t.

We can find the relative acceleration in the usual way of time-differentiating the relative velocity,

$$\bar{a}_1 - \bar{a}_2 = \dot{\bar{v}}_1 - \dot{\bar{v}}_2$$
$$= \dot{\bar{\omega}} \times (\bar{x}_1 - \bar{x}_2) + \bar{\omega} \times (\bar{v}_1 - \bar{v}_2)$$

where $\bar{\alpha} = \dot{\bar{\omega}}$ is the angular acceleration vector.

8.1.4 Fixed-Axis Rotation

All the above is general (rotation about all axes simultaneously). In this class, we often work with planar rotation problems, for which we align the $\bar{E}_z$ basis vector perpendicular to the plane of rotation. In this case,

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where θ is the counterclockwise rotation of the body about $\bar{E}_z$.

It is easy to show that $\dot{Q}Q^T = \dot{\theta}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

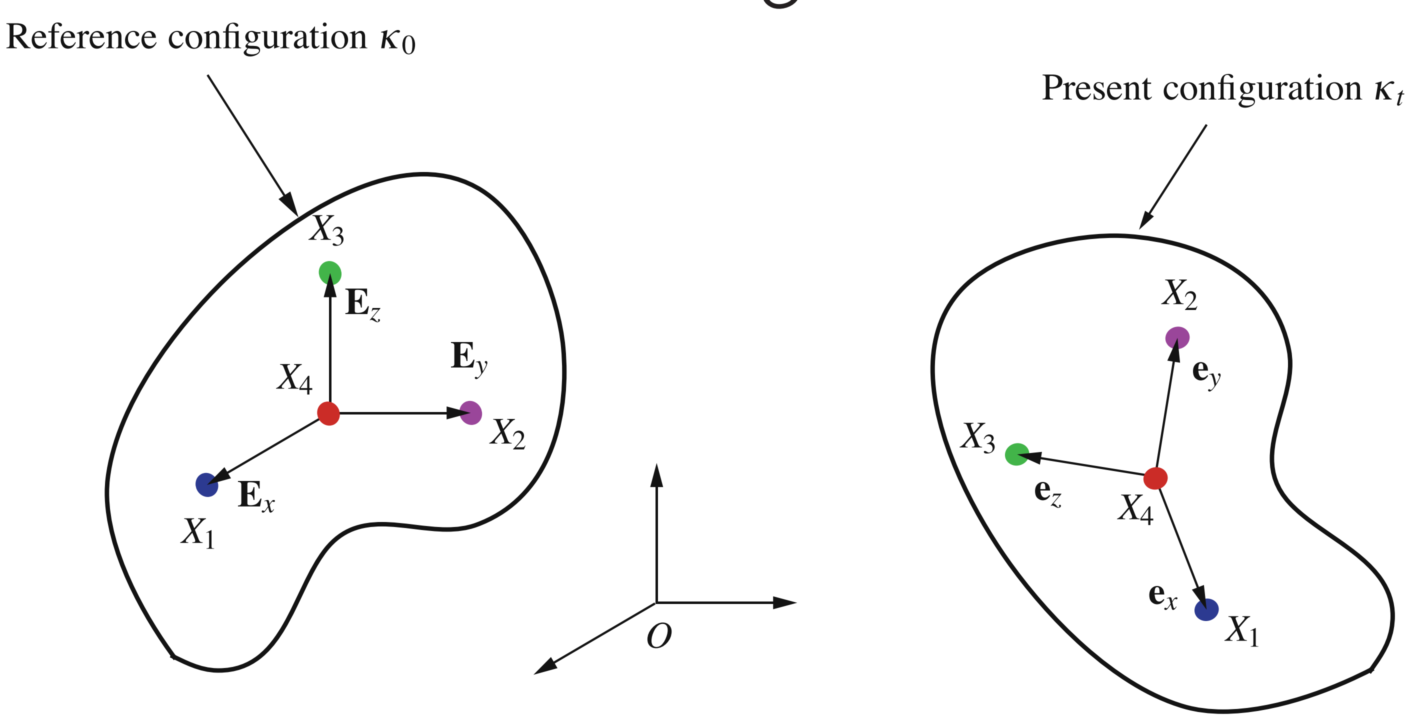
+ .

8.2 Kinematical Relations + a Corotational Basis

Up to this point, we have used a fixed Cartesian basis. We now introduce + explore a convenient basis called a corotational (body-fixed) basis.

8.2.1 The Corotational Basis

We define the corotational basis $(\bar{e}_x, \bar{e}_y, \bar{e}_z)$ that rotates with the body as follows.



First, we choose the four points on the body X_1, X_2, X_3, + X_4 such that

form a fixed, right-handed, Cartesian basis.

Because Q preserves the distance and orientation between points, in all configurations the following vectors form a ("corotational") basis:

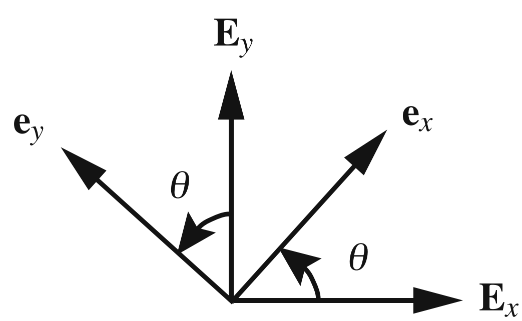
$$\bar{e}_x = \bar{x}_1 - \bar{x}_4 \,, \quad \bar{e}_y = \bar{x}_2 - \bar{x}_4 \,, \quad \bar{e}_z = \bar{x}_3 - \bar{x}_4 \,.$$

We have already derived expressions for relative velocities and accelerations, which can now be used to derive expressions for the time-rate-of-change of $\bar{e}_i$.

and similarly for $\ddot{\bar{e}}_y$ + $\ddot{\bar{e}}_z$. We will use these soon.

8.2.2 The Corotational Basis for a Fixed-Axis

For a fixed axis of rotation, the rotation matrix Q is, in the Cartesian basis,



$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where θ is as in the figure.

This yields the following relations:

$$\begin{bmatrix} \overline{e_x} \\ \overline{e_y} \\ \overline{e_z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{E_x} \\ \overline{E_y} \\ \overline{E_z} \end{bmatrix}.$$

Previously, we showed that, for the fixed-axis case, $\overline{\omega} = \dot{\theta}\,\overline{E_z}$ and $\overline{\alpha} = \ddot{\theta}\,\overline{E_z}$, which we can use to find that

$$\begin{aligned} \dot{\overline{e_x}} &= \overline{\omega} \times \overline{e_x} = \dot{\theta}\,\overline{e_y} \\ \dot{\overline{e_y}} &= \overline{\omega} \times \overline{e_y} = -\dot{\theta}\,\overline{e_x} \\ \dot{\overline{e_z}} &= \overline{\omega} \times \overline{e_z} = \overline{0} \end{aligned}.$$