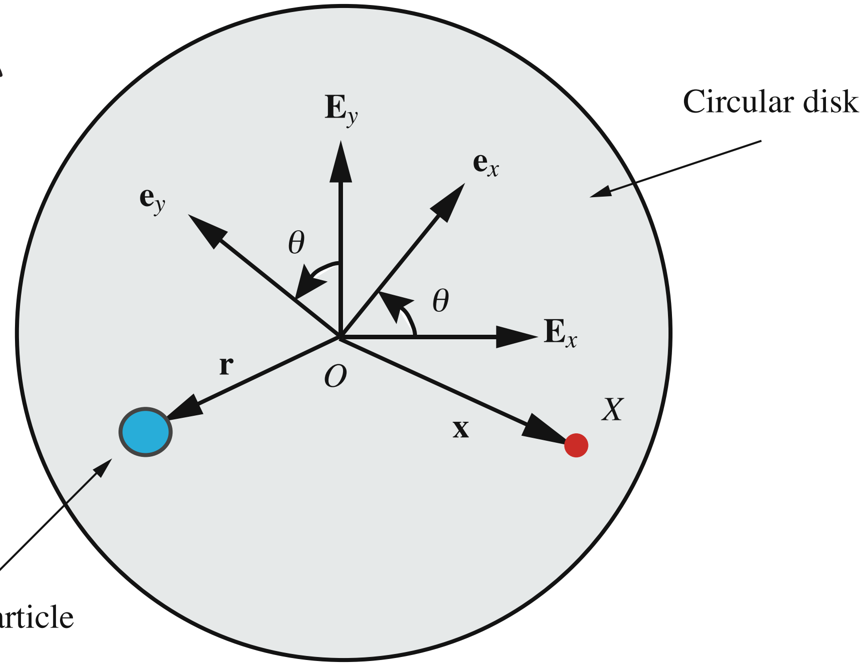
8.2.3 A Particle Moving on a Rigid Body (example)

Consider the disk and mass particle.



The center of the disk is fixed point about which the disk rotates.

The disk rotates about $\bar{E}_z$ with $\bar{\omega} = \dot{\theta}\bar{E}_z = \omega \bar{E}_z$ and angular acceleration $\bar{\alpha} = \ddot{\theta}\bar{E}_z = \alpha \bar{E}_z$.

Suppose the position of the particle is

$$\bar{r} = 10t^2 \bar{e}_x + 20t\,\bar{e}_y$$

and the position of the point X on the disk is

$$\bar{x} = x\bar{e}_x + y\bar{e}_y$$

where x and y are constant scalars.

Find the velocities of the particle and the point X.

Velocity of the particle:

$$\dot{\bar{r}} = 20t\,\bar{e}_x + 10t^2\dot{\bar{e}}_x + 20\bar{e}_y + 20t\,\dot{\bar{e}}_y$$

$$= 20t\,\bar{e}_x + 10t^2\omega\bar{e}_y + 20\bar{e}_y - 20t\omega\bar{e}_x$$

$$= (20t - 20t\omega)\bar{e}_x + (20 + 10t^2\omega)\bar{e}_y\,.$$

Velocity of point $\bar{X}$:

Notice that $\dot{\bar{X}} = \bar{\omega} \times \bar{X} = \det\begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & 0 & \omega \\ x & y & 0 \end{bmatrix}$

because the origin is fixed.

But $\dot{\bar{r}} \neq \bar{\omega} \times \bar{r}$. Why?

Because $\bar{r}$ is the position vector of a particle that is moving independently from the disk.

8.4 Center of Mass and Linear Momentum

In this section we define the center of mass C and the linear momentum $\bar{G}$ of a body.

Let $\mathcal{R}$ denote the region of space occupied by the body in its present configuration. Let $\mathcal{R}_0$ denote the region occupied in the body's reference configuration.

Let the density of the material of the body be $\rho(\bar{x},t)$ in its present configuration and $\rho_0(\bar{X})$ in its reference configuration.

8.4.1 The Center of Mass

The position vectors of the center of mass of the body is

$$\bar{x} = \frac{\int_{\mathcal{R}} \bar{x} \rho \, dv}{\int_{\mathcal{R}} \rho \, dv}$$

for the present configuration and

$$\bar{\bar{X}} = \frac{\int_{\mathcal{R}_0} \bar{X} \rho_0 \, dV}{\int_{\mathcal{R}_0} \rho_0 \, dV} .$$

for the reference configuration.

We assume that mass m is conserved, so

$$dm = \rho_0 \, dV = \rho \, dv$$

$$m = \int_{\mathcal{R}_0} \rho_0 \, dV = \int_{\mathcal{R}} \rho \, dv .$$

So we often write:

$$\boxed{\bar{x} = \frac{1}{m}\int_{R} \bar{x}\,\rho\, dv} \qquad \boxed{\bar{X} = \frac{1}{m}\int_{R_0} \bar{X}\,\rho_0\, dV}.$$

Using the results of Section 8.1.2, we can write an equation relating the center of mass C and another point Y:

$$\bar{x} - \bar{y} = Q \circ (\bar{X} - \bar{Y})$$

Differentiating w.r.t. time to get the relative velocity:

and once more to get the relative acceleration:

$$\bar{a} - \ddot{\bar{y}} = \bar{\alpha} \times (\bar{x} - \bar{y}) + \bar{\omega} \times (\bar{\omega} \times (\bar{x} - \bar{y})).$$

$\bar{v}$ is the velocity of the center of mass

$\bar{a}$ is the acceleration of the center of mass.

8.4.2 The Linear Momentum

Definition: the linear momentum of a rigid body (using the definitions above) is

$$\boxed{\bar{G} = \int_R \bar{v} \rho \, dv \, .}$$

This can be written in the following convenient way:

This result is identical to that for a system of particles.