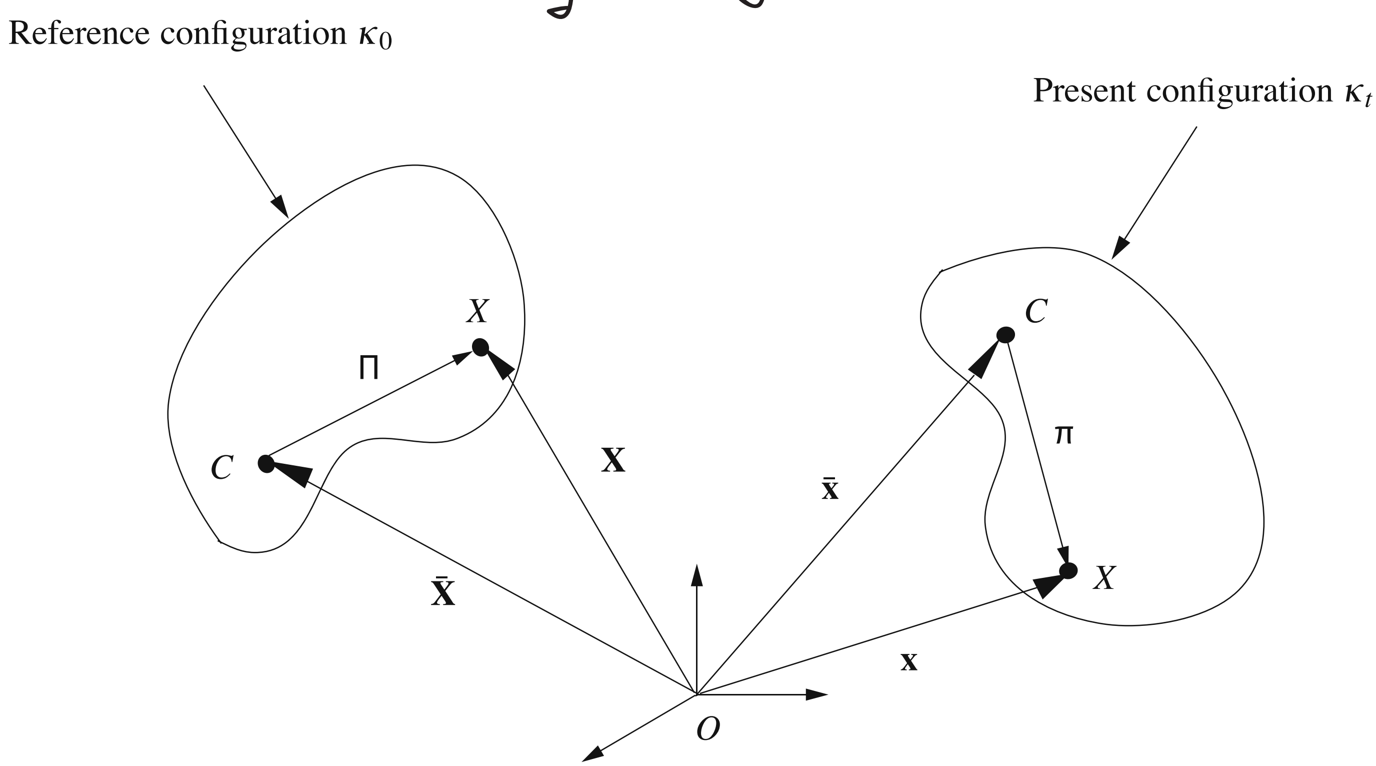
8.7 Angular Momenta

Before we can discuss the balance laws of Chapter 9, we must introduce the angular momentum of a rigid body.



Definition: the angular momentum of $\mathcal{B}$ about its center of mass C is

Definition: the angular momentum of $\mathcal{B}$ about the fixed point O is

These equations can be related as follows.

See O'Reilly p. 153 for an expression for the angular momentum about an arbitrary point.

8.8 Inertia tensor

Expressing the relative position vectors in bases:

Also, we write the angular velocity of the body:

8.8.1 The Inertia Tensor

If we consider the angular momentum of $\mathcal{B}$ about its center of mass $\bar{H}$, we can derive the following important relationship:

where

$\mathbb{I}$ is the inertia tensor, which is written as a matrix in the corotational basis as:

The following components are called moments of inertia:

$$I_{xx} = \int_{R} (\Pi_y^2 + \Pi_z^2)\, \rho\, dv = \int_{R_0} (\Pi_y^2 + \Pi_z^2)\, \rho_0 dV,$$

$$I_{yy} = \int_{R} (\Pi_x^2 + \Pi_z^2)\, \rho\, dv = \int_{R_0} (\Pi_x^2 + \Pi_z^2)\, \rho_0 dV,$$

$$I_{zz} = \int_{R} (\Pi_x^2 + \Pi_y^2)\, \rho\, dv = \int_{R_0} (\Pi_x^2 + \Pi_y^2)\, \rho_0 dV,$$

The remaining components are called products of inertia:

$$I_{xy} = -\int_{R} \Pi_x \Pi_y \rho\, dv = -\int_{R_0} \Pi_x \Pi_y \rho_0 dV,$$

$$I_{xz} = -\int_{R} \Pi_x \Pi_z \rho\, dv = -\int_{R_0} \Pi_x \Pi_z \rho_0 dV,$$

$$I_{yz} = -\int_{R} \Pi_y \Pi_z \rho\, dv = -\int_{R_0} \Pi_y \Pi_z \rho_0 dV.$$

$\mathbf{I}$ is positive-definite, and so its eigenvalues are positive. Its components depend on the basis chosen. If the basis $(\bar{E}_x, \bar{E}_y, \bar{E}_z)$ is chosen such that $\bar{E}_x, \bar{E}_y, + \bar{E}_z$ are the eigenvectors of $\mathbf{I}$, then $\{\bar{E}_x, \bar{E}_y, \bar{E}_z\}$ and $\{\bar{e}_x, \bar{e}_y, \bar{e}_z\}$ are called the principal axes of the body in its reference and present configurations, respectively.

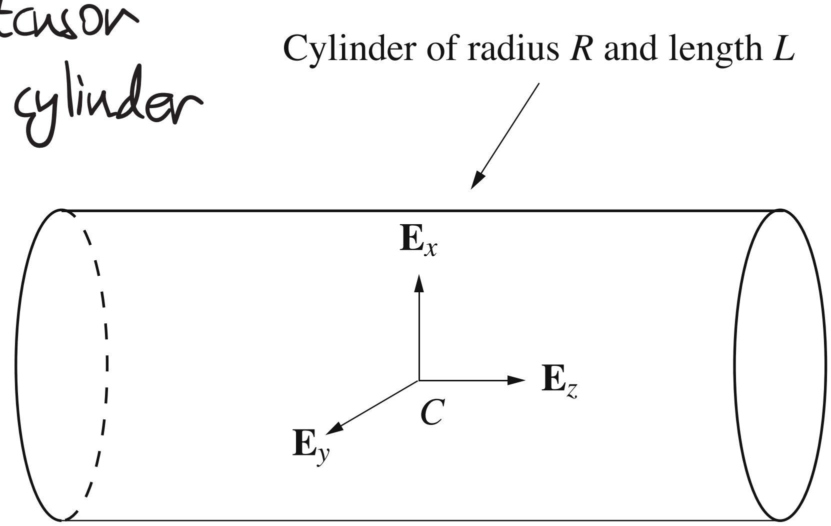
In this case, in the corotational basis,

Therefore, we always try to choose $\bar{E}_x, \bar{E}_y, + \bar{E}_z$ as the principal axes.

8.8.3 A Circular Cylinder (example)

What is the inertia tensor for the homogeneous cylinder of mass m, radius R, and Length L?

Choose the principle axes, as shown.



If $I_{xy} = I_{xz} = I_{yz} = 0$, (and they are) we have correctly chosen the principal axes.

Computing the remaining integrals,

If we set $R=0$, we have the inertia tensor for a "slender rod." With $L=0$, we have the inertia tensor for a "thin disk."

8.8.4 The Parallel Axis Theorem + Practical Notes

The Parallel Axis Theorem is commonly used to find the inertia tensor of a point that is not the center of mass. However, we circumvent the need for it by expressing the angular momentum about an arbitrary point A:

We will use this in Chapter 9. Note that this approach is more general than the PAT, which only applies to points on the body.

To find the moments of inertia, we often refer to a table of common shapes of bodies, like that in the cover of Hibbeler.