Chapter 9: Kinetics of a Rigid Body

9.1 Balance Laws for a Rigid Body

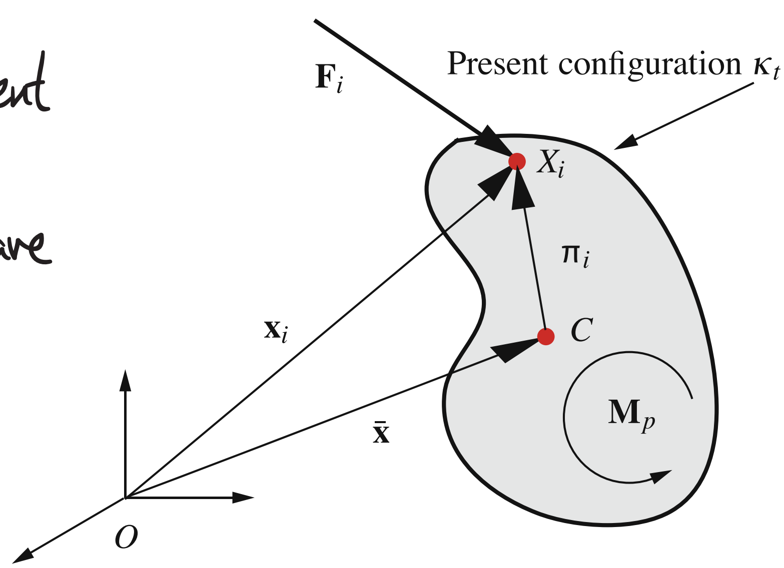
Before we get to the balance laws, we need to discuss forces and moments on a rigid body.

9.1.1 Resultant Forces and Moments

Given a set of n forces $\{\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_i, \ldots, \bar{F}_n\}$ acting on a body $\mathcal{B}$ at material points $\{X_1, \ldots, X_i, \ldots X_n\}$, the resultant force is

Similarly, the resultant moment $\bar{M}_0$ relative to the fixed point O is the sum of individual moments about O acting on $\mathcal{B}$. We denote the resultant moment relative to the center of mass $\bar{M}$.

Given an external moment $\bar{M}_p$ not induced by an $\bar{F}_i$, the resultant moments are



9.1.2 Euler's Laws

Euler's Laws are the momentum balance laws for rigid bodies. The first law is the balance of linear momentum:

(*) .

The second law is the balance of angular momentum:

(**) .

Together, the Euler's Laws give six scalar equations. Another form of the second law is:

(***) .

For rigid bodies with a fixed point (i.e. "pinned"), we typically use (**). For others we often use (***).

Recall from Section 8.8,

$$
\begin{aligned}
\overline{H} &= I\,\overline{\omega} \\
&= (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\,\overline{e_x} \\
&+ (I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\,\overline{e_y} \\
&+ (I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z)\,\overline{e_z}\,.
\end{aligned}
$$

Taking the time-derivative,

$$\dot{\overline{H}} = I\,\dot{\overline{\omega}} = \overset{\circ}{\overline{H}} + \overline{\omega} \times \overline{H}, \quad \text{where}$$

$$\begin{aligned}\overset{\circ}{\overline{H}} = & \left(I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z\right)\overline{e_x} \\ & + \left(I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z\right)\overline{e_y} \\ & + \left(I_{xz}\dot{\omega}_x + I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z\right)\overline{e_z}\end{aligned}$$

is the corotational rate of $\overline{H}$. This gives, in general, a very complicated set of equations.

9.1.3 The Fixed-Axis of Rotation Case

We worked-out the Kinematics in Chapter 8:

$$\overline{e_x} = \cos\theta\,\overline{E_x} + \sin\theta\,\overline{E_y} \quad \Big| \quad \overline{e_y} = \cos\theta\,\overline{E_y} - \sin\theta\,\overline{E_x} \quad \Big| \quad \overline{e_z} = \overline{E_z}$$

$$\dot{\overline{e_x}} = \dot{\theta}\,\overline{e_y} \quad \Big| \quad \dot{\overline{e_y}} = -\dot{\theta}\,\overline{e_x} \quad \Big| \quad \overline{\omega} = \dot{\theta}\,\overline{E_z} = \omega\,\overline{E_z}$$

The angular moment and its time-derivative are:

The kinetics are simply Euler's equations using the above Kinematics:

The first equation gives the motion of the center of mass and reaction forces.

The $\bar{e}_x$ and $\bar{e}_y$ scalar equations from the second equation ($\bar{M} = \dot{\bar{H}}$), give the reaction moment $\bar{M}_c$ that keeps the body rotating about the $\bar{E}_z$-axis ($\bar{M}_c$ is often just $\bar{0}$).

The $\bar{e}_z = \bar{E}_z$ scalar equation from $\bar{M} = \dot{\bar{H}}$ gives the differential equation for $\theta(t)$.

9.1.4 The Four Steps for solving problems! ← this

We follow four steps that are similar to the four we used for particles.

1. Kinematics

Pick: - an origin for the Cartesian basis in a reference configuration. (O)
- a coordinate system to work in $(\bar{E}_x, \bar{E}_y, \bar{E}_z)$
- a corotational basis $(\bar{e}_x, \bar{e}_y, \bar{e}_z)$

Establish expressions for $\bar{H}$ or $\bar{H}_o$, $\bar{x}$, $\bar{v}$, and $\bar{a}$.

2. Forces and moments

Draw a free-body diagram show external forces $\bar{F}_i$ and moments $\bar{M}_p$.

Write what is known about each force and moment.

3. Euler's Laws

Write out the six scalar equations from

$$\bar{F} = m\bar{a} \qquad \Big| \qquad \bar{M} = \dot{\bar{H}} \quad \text{or} \quad \bar{M}_o = \dot{\bar{H}}_o$$

4. Analysis

Solve for what is needed, using the six scalar equations and sometimes additional kinematic equations.