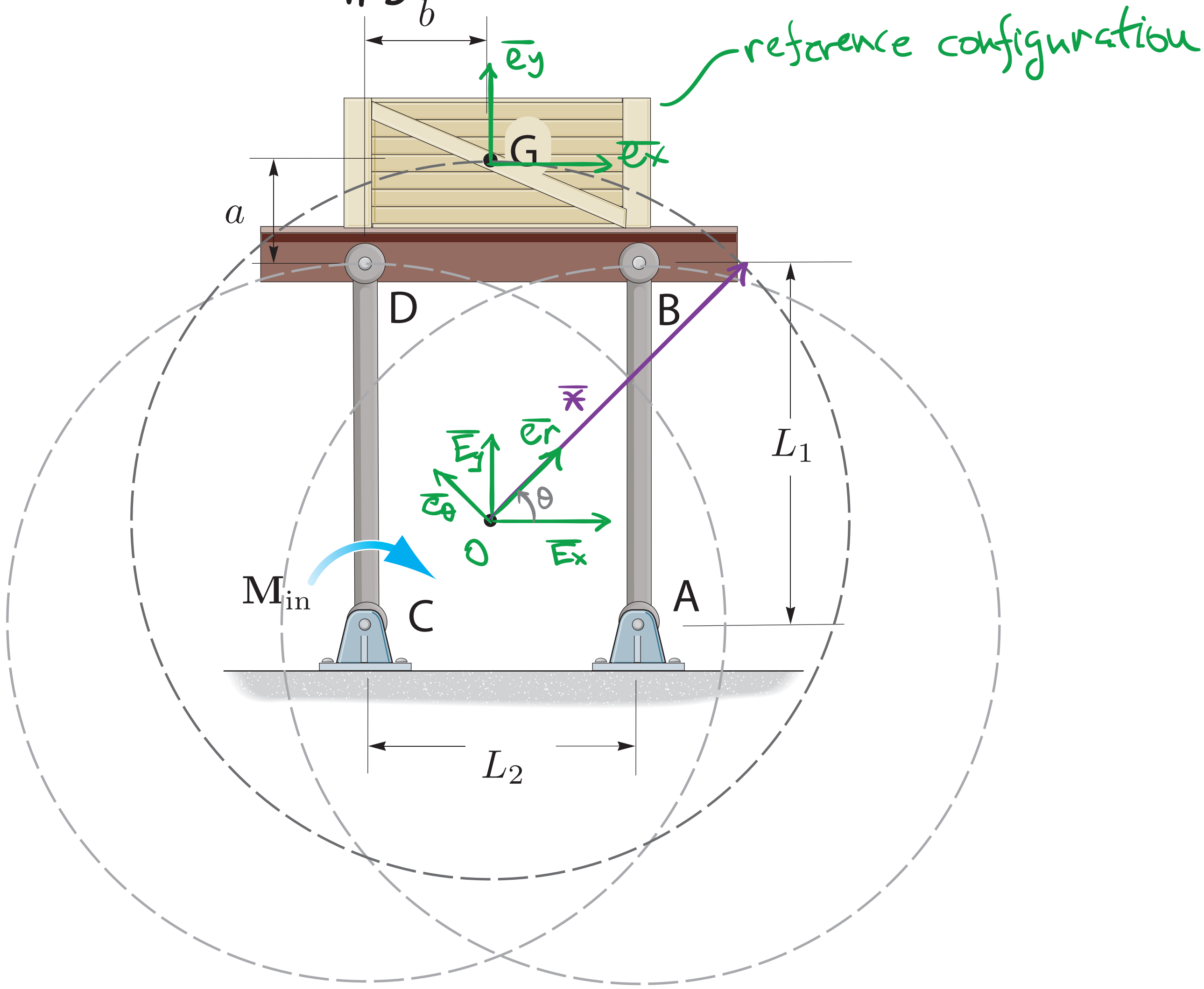
Hibbeler 17-55 (but more-general), the O'Reilly way

Given an applied moment $\overline{M}_{in}$ at C, find the angular acceleration $\ddot{\theta}(\theta)$ of the links and $\overline{T}_1(\theta,\dot{\theta})$ and $\overline{T}_2(\theta,\dot{\theta})$, the pin forces at D and B. Assume massless links and no slipping of the box.



Kinematics

The motion of points G, D, and B follow the circular paths shown. We choose the origin O of our Cartesian coordinate system to be at the center of the circle that G follows. This is convenient because a polar coordinate basis is natural in this case. The corotational basis is colinear with the Cartesian basis because the orientation of the box doesn't change throughout its motion. The position vector of G is

$$\overline{x} = x\overline{E}_x + y\overline{E}_y = r\,\overline{e}_r = L_1\overline{e}_r\,.$$

Either differentiating or using the results of § 2.2,

$$\bar{v} = \dot{r}\,\bar{e}_r + r\dot{\theta}\,\bar{e}_\theta = L_1\dot{\theta}\,\bar{e}_\theta \qquad \text{and}$$

$$\bar{a} = -L_1\dot{\theta}^2\,\bar{e}_r + L_1\ddot{\theta}\,\bar{e}_r \quad .$$

We will also need the position vectors of D and B.

$$\bar{x}_D = \bar{x} - b\,\bar{E}_x - a\bar{E}_y = \bar{x} - b(\cos\theta\,\bar{e}_r - \sin\theta\,\bar{e}_\theta) - a(\sin\theta\,\bar{e}_r + \cos\theta\,\bar{e}_\theta) = (L_1 - b\cos\theta - a\sin\theta)\bar{e}_r + (b\sin\theta - a\cos\theta)\bar{e}_\theta$$

$$\bar{x}_B = \bar{x} + (L_2 - b)\bar{E}_x - a\bar{E}_y = (L_1 + (L_2 - b)\cos\theta - a\sin\theta)\,\bar{e}_r + ((b - L_2)\sin\theta - a\cos\theta)\,\bar{e}_\theta$$

In a moment, we will apply Euler's laws. In anticipation of that, let's compute the angular momentum of the box about O:

$$\bar{H}_O \triangleq \bar{H} + \bar{x}\times\bar{G}$$

Cartesian basis

$$\bar{H} \triangleq I\bar{\omega} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = I_{zz}\overset{0}{\cancel{\omega}}\,\bar{E}_z = 0\bar{E}_z$$

$$\bar{G} \triangleq m\bar{v} = mL_1\dot{\theta}\,\bar{e}_\theta \quad .$$

Combining these expressions:

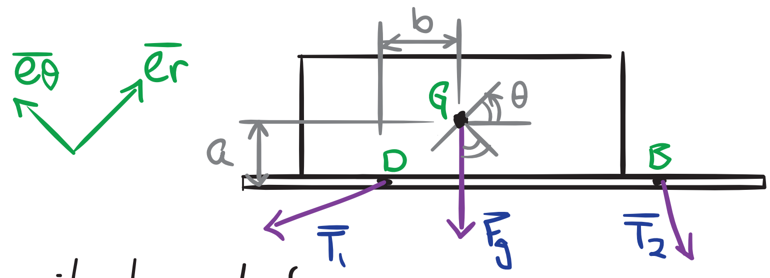
$$\begin{aligned} \bar{H}_O &= L_1\bar{e}_r \times mL_1\dot{\theta}\,\bar{e}_\theta \\ &= \det\begin{bmatrix} \bar{e}_r & \bar{e}_\theta & \bar{E}_z \\ L_1 & 0 & 0 \\ 0 & mL_1\dot{\theta} & 0 \end{bmatrix} \\ &= mL_1^2\dot{\theta}\,\bar{E}_z \quad . \end{aligned}$$

And we'll need the time-derivatives of the linear and angular momenta:

$$\dot{\bar{G}} = m\bar{a} = m\left(-L_1\dot{\theta}^2\,\bar{e}_r + L_1\ddot{\theta}\,\bar{e}_r\right)$$

$$\dot{\bar{H}}_O = mL_1^2\,\ddot{\theta}\,\bar{E}_z \quad .$$

Forces + Moments on the box



The bars exert the forces:

$$\overline{T_1} = T_{1r}\,\overline{e_r} + T_{1\theta}\,\overline{e_\theta} \quad \&$$
$$\overline{T_2} = T_{2r}\,\overline{e_r} + T_{2\theta}\,\overline{e_\theta} \ .$$

The gravitational force is:

$$\overline{F_g} = -mg\sin\theta\,\overline{e_r} - mg\cos\theta\,\overline{e_\theta}$$

Resultant force: $\overline{F} = \overline{F_g} + \overline{T_1} + \overline{T_2} = (-mg\sin\theta + T_{1r} + T_{2r})\overline{e_r} + (-mg\cos\theta + T_{1\theta} + T_{2\theta})\overline{e_\theta}$.

Moments about O:

$$\overline{M_g} = \overline{x} \times \overline{F_g} = L_1\overline{e_r} \times (-mg\sin\theta\,\overline{e_r} - mg\cos\theta\,\overline{e_\theta}) = -mgL_1\cos\theta\,\overline{E_z}$$

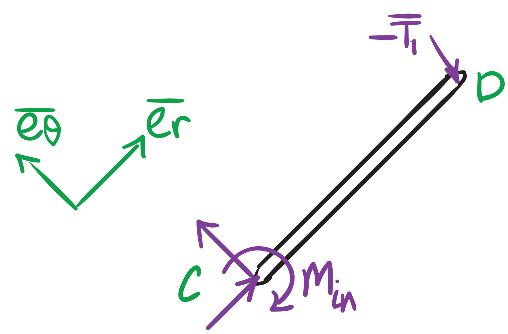
$$\overline{M_1} = \overline{x_D} \times \overline{T_1} = \big((aT_{1r} - bT_{2r})\cos\theta - (bT_{1r} + aT_{2r})\sin\theta\big)E_z$$

$$\overline{M_2} = \overline{x_B} \times \overline{T_2} = \big(L_1T_{2\theta} + (aT_{1\theta} + (L_2 - b)T_{2\theta})\cos\theta - (bT_{1\theta} - L_2T_{1\theta} + aT_{2\theta})\sin\theta\big)\overline{E_z}.$$

Resultant moment: $\overline{M_O} = \overline{M_g} + \overline{M_1} + \overline{M_2}$. $\overline{M}_{in}$ doesn't enter because it's applied to the link CD, not the box.

Forces + Moments on Link CD

The resultant moment about C is:



$$\overline{M_C} = \overline{M_{in}} + L_1\overline{e_r} \times (-\overline{T_1}) = \overline{M_{in}} - L_1T_{1\theta}\,\overline{E_z}.$$

Introducing the resultant force doesn't help us because we don't know or care about the reaction at C.

Euler's Laws on the links

The link CD is massless, so its angular momentum is zero and

$$\overline{M_C} = \overline{0} \Rightarrow \overline{M_{in}} - L_1T_{1\theta}\overline{E_z} = \overline{0} \Rightarrow T_{1\theta} = \frac{1}{L_1}\overline{M_{in}}\cdot\overline{E_z} .$$ Similarly: $T_{2\theta} = 0$.

Euler's Laws on the box

First Law: $\bar{F} = \dot{\bar{G}} = m\bar{a}$ which, written in the polar coord's, is

$$\begin{bmatrix} -mg\sin\theta + T_{1r} + T_{2r} \\ -mg\cos\theta + T_{1\theta} + T_{2\theta} \end{bmatrix} = \begin{bmatrix} m(-L_1\dot{\theta}^2) \\ mL_1\ddot{\theta} \end{bmatrix}$$

Second Law: $\bar{M}_0 = \dot{\bar{H}}_0 \Rightarrow \bar{M}_g + \bar{M}_1 + \bar{M}_2 = mL_1^2\ddot{\theta}\bar{E}_z$

Analysis

The second scalar equation from the First Law gives:

$$\begin{aligned}\ddot{\theta} &= \frac{1}{mL_2}(-mg\cos\theta + T_{1\theta} + \cancel{T_{2\theta}}^{\,0}) \\ &= \frac{1}{mL_1}\left(-mg\cos\theta + \frac{1}{L_1}M_{in}\right) \\ &= \frac{1}{mL_1^2}(-mgL_1\cos\theta + M_{in}) \quad \longleftarrow \text{ANS}\end{aligned}$$

We know $T_{1\theta} = \frac{1}{L_1}\bar{M}_{in}\cdot\bar{E}_z$ and $T_{2\theta} = 0$ and $\ddot{\theta}(\theta)$. We still want $T_{1r}(\theta,\dot{\theta})$ and $T_{2r}(\theta,\dot{\theta})$, so we need two equations with T_{1r}, T_{2r}, θ, + $\dot{\theta}$ the only unknowns.

The first scalar equation of the First Law is one. The Second Law gives only one (nontrivial) scalar equation. They are linear and easy to solve:

$\bar{T}_1(\theta,\dot{\theta}) = T_{1r}\bar{e}_r + T_{1\theta}\bar{e}_\theta$ where $T_{1r}(\theta,\dot{\theta}) =$ $\longleftarrow$ ANS

$$\frac{mL_1^3\ddot{\theta} + (b-L_2)M_{in}\sin\theta + agL_1m\sin^2\theta + (mgL_1^2 - aM_{in} + bL_1mg\sin\theta)\cos\theta - mL_1^2(b\cos\theta + a\sin\theta)\dot{\theta}^2}{(a+b)L_1\cos\theta + (a-b)L_1\sin\theta}$$

$\bar{T}_2(\theta,\dot{\theta}) = T_{2r}\bar{e}_r + T_{2\theta}\bar{e}_\theta$ where $T_{2r}(\theta,\dot{\theta}) =$ $\longleftarrow$ ANS

$$\frac{-mL_1^3\ddot{\theta} - (b-L_2)M_{in}\sin\theta - bgL_1m\sin^2\theta + (-mgL_1^2 - aM_{in} + aL_1mg\sin\theta)\cos\theta + mL_1^2(-a\cos\theta + b\sin\theta)\dot{\theta}^2}{(a+b)L_1\cos\theta + (a-b)L_1\sin\theta}$$

Note that there was a more-convenient point about which to apply Euler's Laws: G. Because the box has no rotation $\bar{\alpha} = \bar{0}$, and $\bar{M} = 0$. This simplifies the solution for $\bar{T}_1(\theta,\dot{\theta})$ and $\bar{T}_2(\theta,\dot{\theta})$, mostly because the position vectors in the moment equations are easier.