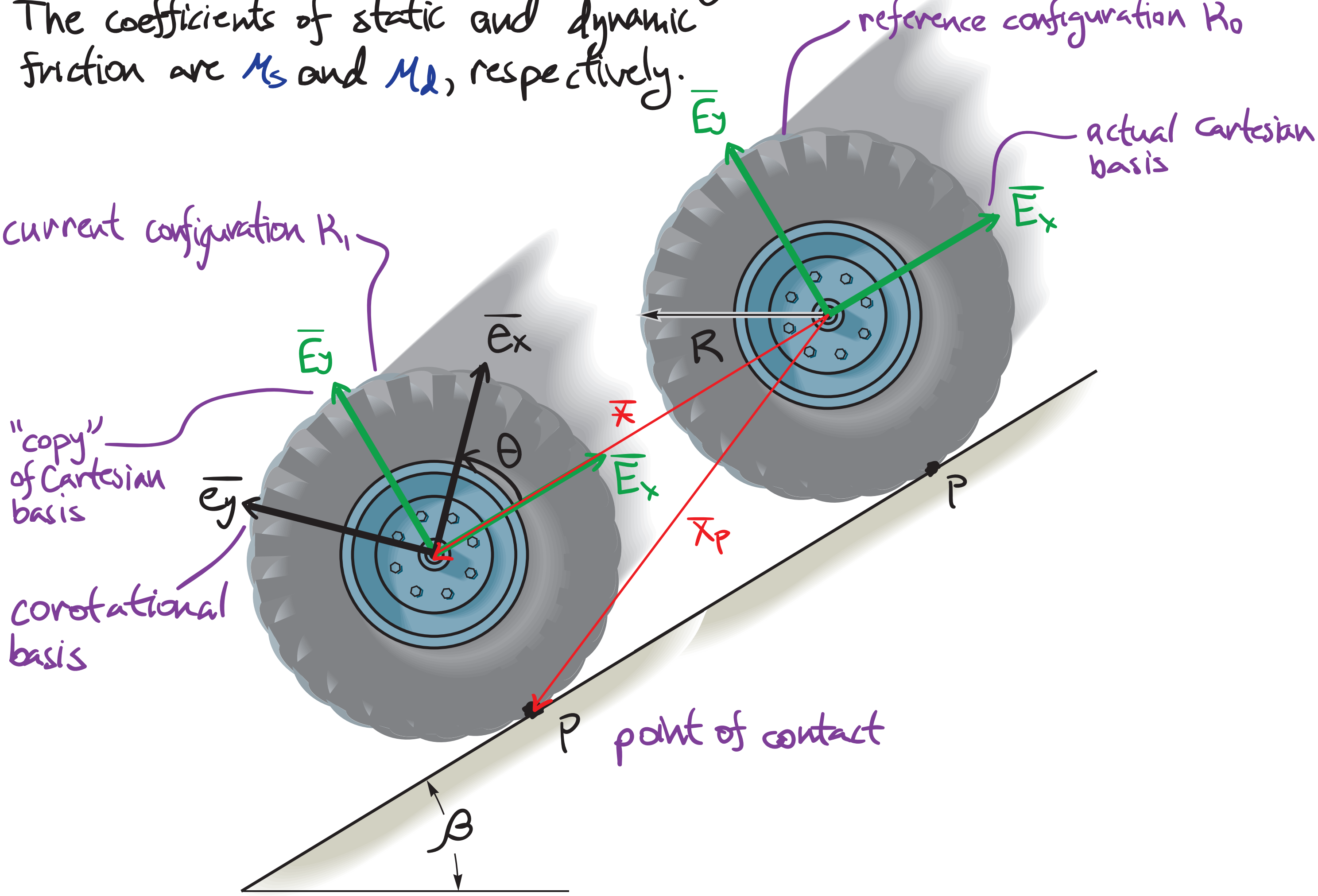
Example Based-On Hibbeler 17-94

Given the wheel with moment of inertia about its axle I_{zz} rolling down the incline, find the angular acceleration of the wheel if it doesn't slip and the range of β for which this is valid. The coefficients of static and dynamic friction are μ_s and μ_d, respectively.



Kinematics

Since the wheel isn't pinned, we're probably going to be interested primarily in the motion of the center of mass. We place the origin of the Cartesian basis at the center of mass in the reference configuration κ_0. The corotational basis has angle θ with respect to the Cartesian basis at some later ("present") configuration κ_1.

Position of the center of mass:

Velocity of the center of mass:

Acceleration of the center of mass:

Since we will have a moment equation (we have a rigid body, after all, so there's always a moment equation), we will want to know the angular velocity and acceleration.

Angular velocity: $\quad \bar{\omega} = \omega \bar{E}_z = \dot{\theta} \bar{E}_z$.

Angular acceleration: $\quad \bar{\alpha} = \alpha \bar{E}_z = \ddot{\theta} \bar{E}_z$.

When the wheel is rolling without slipping, we can relate the translation and rotation of the wheel with a convenient constraint. Even if the wheel is slipping, we can relate the velocities of the center of mass and the instantaneous point of contact P with the surface:

If there is no slipping, $\bar{v}_p = \bar{0}$, + we get the familiar kinematic constraint.

If there is slipping, but no loss of contact, we get

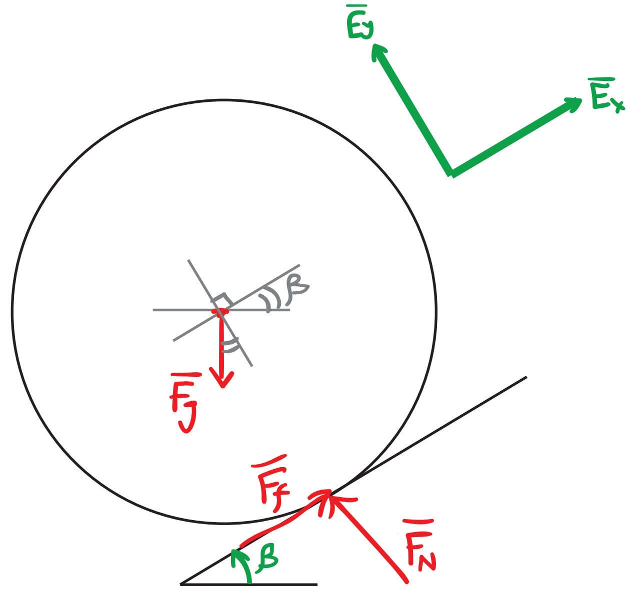
$$\bar{v} = (v_p - R\omega)\bar{E}_x \, .$$

Finally, we can write the linear momentum and the angular momentum about the center of mass, and their time-derivatives:

$$\bar{G} = m\bar{v}$$
$$\dot{\bar{G}} = m\bar{a}$$

$$\bar{H} = I\bar{\omega} = I_{zz}\,\omega\,\bar{E}_z$$
$$\dot{\bar{H}} = I_{zz}\,\alpha\,\bar{E}_{zz} \, .$$

Forces and Moments



Gravity:

Normal:

Friction:

Moment of friction:

Unpacking the friction force, we have two situations: (1) when the magnitude of the friction force

(no slipping)

and (2) when the friction force is

$$\bar{F}_f = -\mu_d \|\bar{F}_N\| \frac{\bar{v}_P}{\|\bar{v}_P\|}. \qquad \text{(slipping)}$$

Since we are trying to find the conditions for no slipping, we will use the first inequality.

Euler's Laws

Assuming no slipping (which we will then have to check), the first Law gives:

The second Law (about the center of mass) gives:

Analysis

We have three unknowns: F_f, F_N, and α. The first law gave us two (nontrivial) scalar equations and the second law gave us one. They are linear, so they are easy to solve.

But we assumed that the wheel rolled without slipping, so we need to show the conditions under which this is true: