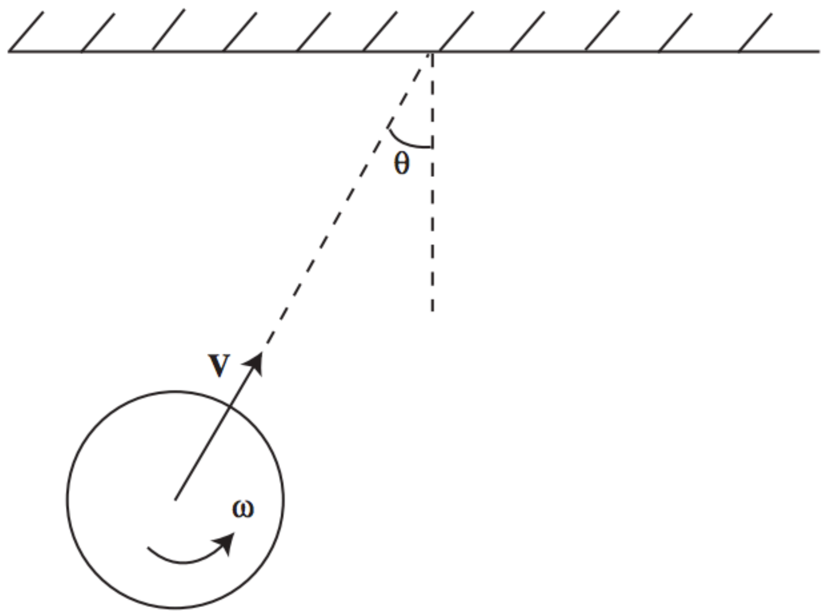
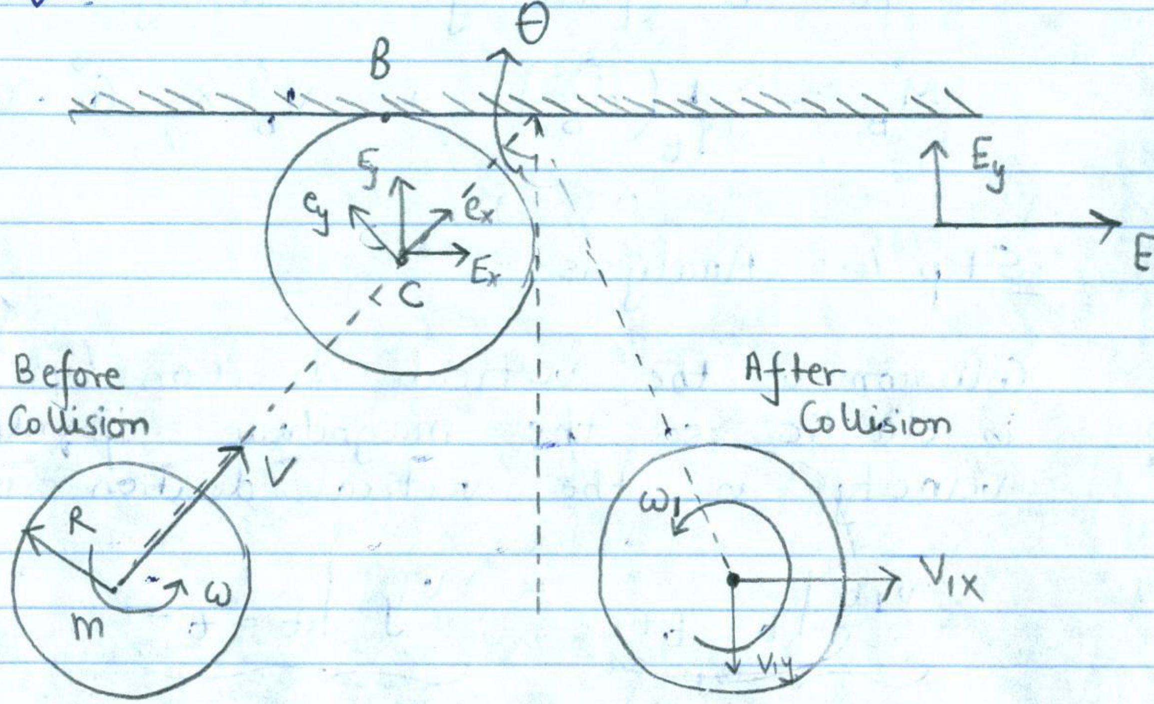
ME 230: Kinematics and Dynamics
Spring 2014 – Section AD

Final Exam Review: Rigid Body Dynamics Practice Problem

1. A rigid uniform flat disk of mass m, and radius R is moving in the plane towards a wall with linear velocity **V** while rotating with angular velocity ω, as shown in the figure below. Assuming that the collision in the normal direction is elastic and that no slip occurs at the wall, find the velocity (of the center of mass) of the disk after it collides with the wall.



Step 1: Pick an origin, a co-ordinate system, a corotational basis, establish equation for $\underline{a}$, $\underline{v}$.



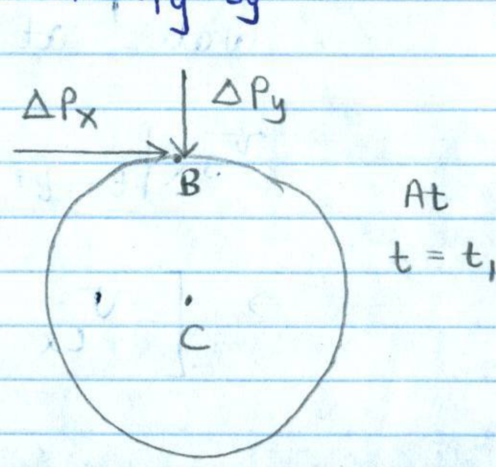
We assume that the disk collides with the wall at point B at time $t = t_1$.

Let the angular velocity before impact be ω, and the angular velocity after impact be ω_1.

The velocity of the center of mass after impact is: $\vec{v} = v_{1x}\vec{E_x} - v_{1y}\vec{E_y}$

Step 2: FBD

Assume an impulse acts on the disk $(\Delta P_x, \Delta P_y)$ at $t = t_1$.



Step 3 : Write out $\vec{M}_O = \dot{\vec{H}}$,

For point B, (point of contact of collision),

$$\vec{M}_B = \frac{d}{dt}\left(\vec{H}_B\right) + \vec{v}_B \times \vec{G} = 0 \quad \text{—(i)}$$

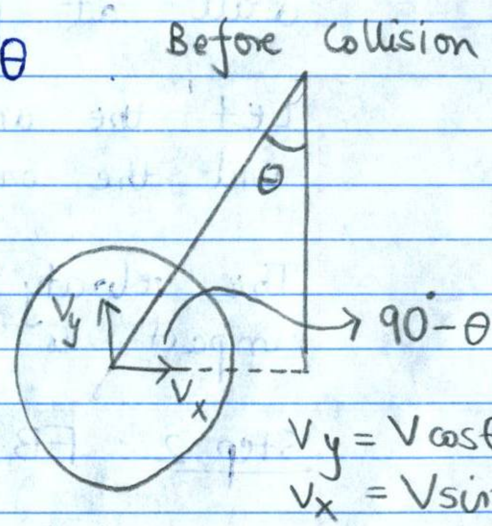
Step 4 : Analysis :

Collision in the vertical direction (y-direction) is elastic so the magnitude of the velocity in the vertical direction is conserved

$$\underbrace{v_{1y}\big|_{t=t^+}}_{\text{After collision}} = \underbrace{v_y\big|_{t=t^-}}_{\text{Before collision}}$$

$$\Rightarrow v_{1y}\big|_{t=t^+} = v\cos\theta$$

$$\Rightarrow \boxed{v_{1y} = v\cos\theta} \quad \text{—(ii)}$$



$V_y = V\cos\theta$
$V_x = V\sin\theta$

No slip occurs at the wall, at point B,

$$v_{Bx}\big|_{t=t^+} = 0$$

$$\Rightarrow \left[v_{cx} + \left(\omega E_z \times r_{B/c}\right)\right]_{t=t^+} = 0$$

$$\Rightarrow \left[v_{cx} + \left(\omega E_z \times R E_y \right) \right]_{t=t^+} = 0$$

$$\Rightarrow v_{1x} - \omega_1 R = 0$$

$$\Rightarrow \boxed{v_{1x} = R\omega_1} \quad \text{--- (iii)}$$

From equation (i),

$$\vec{M}_B = \frac{d}{dt}\left(\vec{H}_B\right) + \cancelto{0}{\vec{v}_B \times \vec{G}} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\vec{H}_B\right) = 0$$

$$\Rightarrow \vec{H}_B\Big|_{t=t^-} = \vec{H}_B\Big|_{t=t^+}$$

Now, $\vec{H}_B = \vec{H}_C + \vec{r}_{B/C} \times \vec{G}$

$$\Rightarrow \underbrace{\frac{1}{2} m R^2 \omega + m V R \sin\theta}_{\text{Angular momentum before collision}} = \underbrace{\frac{1}{2} m R^2 \omega_1 + m R v_{1x}}_{\text{Angular momentum after collision}}$$

Using $\omega_1 = v_{1x}/R$, (from (iii)),

$$\Rightarrow v_{1x} = \frac{2}{3} V \sin\theta + \frac{1}{3} R\omega \quad \text{--- (iv)}$$

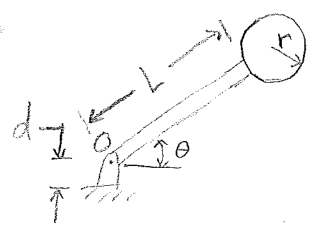
Thus, the velocity of the center of mass after collision is:

$$v_1 = \sqrt{v_{1x}^2 + v_{1y}^2}$$

From (ii) and (iv),

$$\boxed{v_1 = \sqrt{\left(\frac{2}{3} v \sin\theta + \frac{1}{3} R\omega\right)^2 + \left(v \cos\theta\right)^2}}$$

19-56 Hibbeler



The pendulum consists of a sphere and rod with masses m_s and m_r, respectively. Determine the angle θ of rebound after the sphere strikes the floor. The floor has a coefficient of restitution e $(0<e<1)$.

Step 1 : - Origin will be at O.
- We can work with $\hat{E}_x$ and $\hat{E}_y$ for this problem.
- the corotational basis can be $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ s.t.

$$\hat{e}_x = \cos\theta\,\hat{E}_x + \sin\theta\,\hat{E}_y$$

$$\hat{e}_y = -\sin\theta\,\hat{E}_x + \cos\theta\,\hat{E}_y$$

(polar coordinate system)

$$\hat{e}_z = \hat{E}_z \quad \text{(vertical axis unchanging)}$$

(O'Reilly) $$\vec{H}_O = \sum_{k=1}^{K} \vec{r}_K \times {}_p m_k \dot{r}_k + \sum_{n=1}^{N} \left(H_n + (\vec{x}_n \times {}_r m_n \dot{\vec{x}}_n) \right)$$

2 rigid body bodies (sphere and rod)

$$\vec{H}_O = \left[I_{zzO,\,rod}\,\dot{\theta} + I_{zzO,\,sphere}\,\dot{\theta} \right] \hat{e}_z$$

$$I_{zzO,\,rod} = \frac{1}{12} m_r L^2 + m_r \left(\frac{L}{2}\right)^2 = \frac{1}{3} m_r L^2$$

$$I_{zzO,\,sphere} = \frac{1}{2} m_s r^2 + m_s (L+r)^2$$

(parallel axis theorem)

$$\vec{H}_O = \left(\frac{1}{3} m_r L^2 + \frac{1}{2} m_s r^2 + m_s (L+r)^2 \right) \dot{\theta}\,\hat{e}_z$$

$$\vec{X}_{rod} = \frac{L}{2}\hat{e}_x + (0)\hat{e}_y = \frac{L}{2}\cos\theta\,\hat{E}_x + \frac{L}{2}\sin\theta\,\hat{E}_y$$

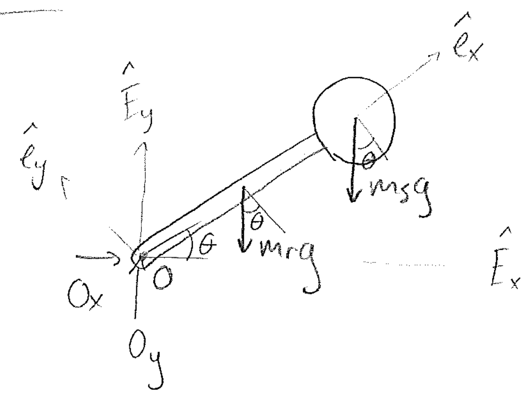
$$\vec{V}_{rod} = -\frac{L}{2}\sin\theta\,\dot{\theta}\,\hat{E}_x + \frac{L}{2}\cos\theta\,\dot{\theta}\,\hat{E}_y$$

$$\vec{X}_{sphere} = (L+r)\hat{e}_x + (0)\hat{e}_y = (L+r)\cos\theta\,\hat{E}_x + (L+r)\sin\theta\,\hat{E}_y$$

$$\vec{V}_{sphere} = -(L+r)\sin\theta\,\dot{\theta}\,\hat{E}_x + (L+r)\cos\theta\,\dot{\theta}\,\hat{E}_y$$

19-56 Hibbeler (continued)

Step. 2



Step 3

$$\sum \vec{F} = m\vec{a}\ ,\quad \sum \vec{M} = \dot{\vec{H}}\ ,\quad \sum \vec{M}_o = \dot{\vec{H}}_o$$

I chose to use energy, but this problem can be solved using $\sum \vec{M}_o = \dot{\vec{H}}_o$. Point O is fixed, and the sum of moments about this point would eliminate O_x, O_y from our expression. $\sum \vec{F} = m\vec{a}$ would not be sufficient in this case.

$$\sum \vec{M}_o = -\left(\cos\theta\left(\tfrac{L}{2}\right) m_r g + \cos\theta\,(L+r)\, m_s g\right)\hat{e}_z$$

$$\dot{\vec{H}}_o = \left(\tfrac{1}{3} m_r L^2 + \tfrac{1}{2} m_s r^2 + m_s (L+r)^2\right)\ddot{\theta}\,\hat{e}_z$$

$$\ddot{\theta} = \frac{-\cos\theta\left(\frac{L m_r g}{2} + (L+r) m_s g\right)}{\left(\frac{1}{3} m_r L^2 + m_s\left(\frac{3}{2} r^2 + L^2 + 2Lr\right)\right)}$$

see the following pages for the solution found using Work - Energy Principles.

19-56 Hibbeler (continued)

Kinetic Energy : $T = \sum_{i=1}^{2} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i + \frac{1}{2} \vec{H} \cdot \vec{\omega}$ (O'Reilly)

$$T = \frac{1}{2}\left(\frac{1}{3} m_r L^2 + \frac{1}{2} m_s r^2 + m_s (L+r)^2\right)\dot{\theta}^2$$

Forces : $\sum \vec{F} = -m_r g \hat{E}_y - m_s g \hat{E}_y$

(There are also forces at O, but we will not need these for analysis).

so $\dot{T} = \sum_{k=1}^{n} \vec{F}_k \cdot \vec{v}_k$

(1) $\dot{T} = -\left(\frac{L}{2}\cos\theta\dot{\theta}\right)(m_r g) - (L+r)(\cos\theta\dot{\theta})(m_s g)$

If we use this expression, the $\dot{\theta}$'s will cancel and we will get an expression for $\ddot{\theta}$!

(2) $\frac{d}{dt}(T) = \frac{1}{2}\left(\frac{1}{3} m L^2 + \frac{1}{2} m r^2 + m(L+r)^2\right) 2\dot{\theta}\ddot{\theta}$

equating (1) and (2) gives

$$\left(\frac{1}{3} m_r L^2 + \frac{1}{2} m_s r^2 + m_s (L+r)^2\right)\dot{\theta}\ddot{\theta} = -\cos\theta\dot{\theta}\left(\frac{L m_r g}{2} + (L+r) m_s g\right)$$

$$\left(\frac{1}{3} m_r L^2 + m_s\left(\frac{3}{2} r^2 + L^2 + 2Lr\right)\right)\ddot{\theta} = -\cos\theta\left(\frac{L m_r g}{2} + (L+r) m_s g\right)$$

same as found by $\sum \vec{M}_O = \dot{\vec{H}}_O$

$$\ddot{\theta} = \frac{-\cos\theta\left(\frac{L m_r g}{2} + (L+r) m_s g\right)}{\left(\frac{1}{3} m_r L^2 + m_s\left(\frac{3}{2} r^2 + L^2 + 2Lr\right)\right)}$$

Step 4

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} \cdot \frac{d\theta}{d\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

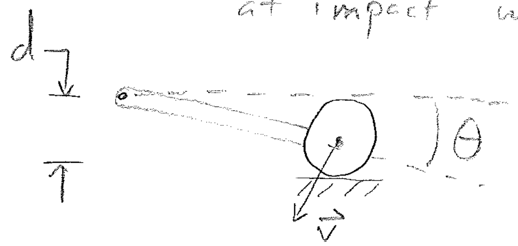
$$\int_0^{\dot{\theta}} \Omega \, d\Omega = \int_{\pi/2}^{\theta} \frac{-\left(\frac{L m_r g}{2} + (L+r) m_s g\right)\cos\alpha}{\left(\frac{1}{3} m_r L^2 + m_s\left(\frac{3}{2} r^2 + L^2 + 2Lr\right)\right)} d\alpha$$

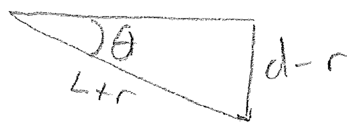
$$\frac{1}{2}\dot{\theta}^2 = \frac{-\left(\frac{L m_r g}{2} + (L+r) m_s g\right)(\sin\theta - 1)}{\left(\frac{1}{3} m_r L^2 + m_s\left(\frac{3}{2} r^2 + L^2 + 2Lr\right)\right)}$$

19-56 Hibbeler (continued)

$$\dot{\theta} = \left[\frac{2(1-\sin\theta)\left(\frac{L m_r g}{2} + (L+r) m_s g\right)}{\left(\frac{1}{3} m_r L^2 + m_s \left(\frac{3}{2} r^2 + L^2 + 2Lr\right)\right)} \right]^{1/2} \qquad (3)$$

at impact with ground:





$$\sin\theta = \frac{d-r}{L+r}$$

$$\theta_I = \sin^{-1}\left(\frac{d-r}{L+r}\right) \qquad (4)$$

$$e = \frac{\vec{V}_2{}' \cdot \hat{n} - \vec{V}_1{}' \cdot \hat{n}}{\vec{V}_1 \cdot \hat{n} - \vec{V}_2 \cdot \hat{n}}$$

velocity at impact:

$$V_{sphere} = -(L+r)\sin\theta_I \dot{\theta}_I \hat{E}_x + (L+r)\cos\theta_I \dot{\theta}_I \hat{E}_y$$

we know θ, $\dot{\theta}$ from (3) + (4) ∴ we know $\vec{V}_{sphere}$

$\hat{n} = \hat{E}_y$, $\vec{V}_1 = 0$ (ground at rest)

$$e = \frac{V_{2,y}'}{-V_{2,y}}$$

$$-V_{2,y} = -(L+r)\cos\theta_I \dot{\theta}_I$$

$$V_{2,y}' = (L+r)\cos\theta_I \dot{\theta}'$$

$e = \dfrac{\dot{\theta}'}{\dot{\theta}}$ or $\dot{\theta}' = -e\,\dot{\theta}_I$ (angular velocity of rigid bodies after impact with ground)

we can find the rebound angle from conservation of energy.

19-56 Hibbeler (continued)

$$T_1 + V_1 = \cancel{T_2}^{\,0} + V_2$$ where "1" is after impact and "2" is highest angle reached.

$$T_1 = \frac{1}{2}\vec{H}' \cdot \vec{\omega}' = \frac{1}{2}\left(\frac{1}{3}m_r L^2 + \frac{1}{2}m_s r^2 + m_s(L+r)^2\right)\dot{\theta}'^2$$

$$T_1 = \frac{1}{2}\left(\frac{1}{3}m_r L^2 + \frac{1}{2}m_s r^2 + m_s(L+r)^2\right)e^2\dot{\theta}_I^2$$

$$V_2 - V_1 = m_r g(X_{rod,y,2} - X_{rod,y,1}) + m_s g(X_{sphere,y,2} - X_{sphere,y,1})$$

$$X_{rod,y,1} = \frac{L}{2}\sin\theta_I \qquad X_{sphere,y,1} = (L+r)\sin\theta_I$$

$$X_{rod,y,2} = \frac{L}{2}\sin\theta_f \qquad X_{sphere,y,2} = (L+r)\sin\theta_f$$

$$V_2 - V_1 = m_r g\left(\frac{L}{2}\right)(\sin\theta_f - \sin\theta_I) + m_s g(L+r)(\sin\theta_f - \sin\theta_I)$$

$$\underbrace{\frac{1}{2}\left(\frac{1}{3}m_r L^2 + \frac{1}{2}m_s r^2 + m_s(L+r)^2\right)e^2\dot{\theta}_I^2}_{\equiv A} = \frac{m_r g L}{2}(\sin\theta_f - \sin\theta_I) + m_s g(L+r)(\sin\theta_f - \sin\theta_I)$$

$$A = \sin\theta_f\underbrace{\left(\frac{m_r g L}{2} + m_s g(L+r)\right)}_{\equiv B} - \sin\theta_I\left(\frac{m_r g L}{2} + m_s g(L+r)\right)$$

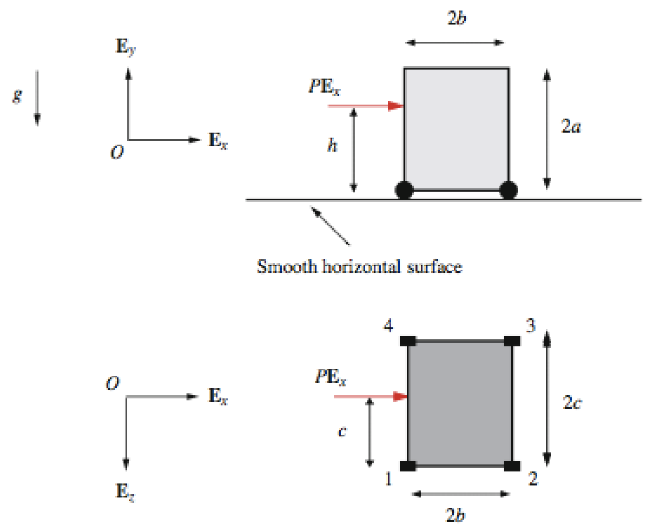
$$\sin\theta_f = \frac{A}{B} + \sin\theta_I$$

$$\boxed{\theta_f = \sin^{-1}\left[\frac{A}{B} + \sin\theta_I\right]}$$

Section AA
ME 230
Finals review

ME230 Kinematics and Dynamics
Rigid Body Dynamics

1) O'Reilly 9.3 Consider the cart shown below,



Suppose that the applied force $\mathbf{P} = 0$, but the front wheels are driven. The driving force on the respective front wheels is assumed to be

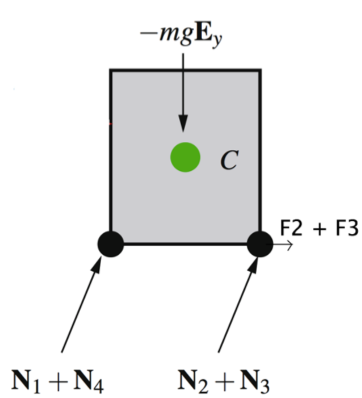
$$\mathbf{F}_2 = \mu N_{2,y}\mathbf{E}_x, \qquad \mathbf{F}_3 = \mu N_{3,y}\mathbf{E}_x$$

where μ is a constant. Calculate the resulting acceleration vector of the center of mass of the cart.

As for the example shown in the book, we first start by with the kinematics and choose a Cartesian coordinate system to describe the position of the center of mass, $\mathbf{x}_C$,

$$\mathbf{x}_C = x\hat{E}_x + y_0\hat{E}_y + z_0\hat{E}_z,$$

where y_0 and z_0 are constants because we consider the case where all 4 wheels of the cart never leave the ground.



The resultant force acting on the system is given by,

$$\begin{aligned}\mathbf{F} = &\ \mu N_{2,y}\hat{E}_x + \mu N_{3,y}\hat{E}_x - mg\hat{E}_y \\ &+ (N_{1,y} + N_{2,y} + N_{3,y} + N_{4,y})\hat{E}_y \\ &+ (N_{1,z} + N_{2,z} + N_{3,z} + N_{4,z})\hat{E}_z\end{aligned}$$

Resultant moment about center of mass is given by,

$$\begin{aligned}\mathbf{M} = &\ \mu N_{2,y}a\hat{E}_z + \mu N_{3,y}a\hat{E}_z \\ &+ c(-N_{1,y} - N_{2,y} + N_{3,y} + N_{4,y})\hat{E}_x \\ &- a(N_{1,z} + N_{2,z} + N_{3,z} + N_{4,z})\hat{E}_x \\ &+ b(N_{1,z} - N_{2,z} - N_{3,z} + N_{4,z}))\hat{E}_y \\ &+ b(-N_{1,y} + N_{2,y} + N_{3,y} - N_{4,y})\hat{E}_z\end{aligned}$$

Use force and moment balance for the cart rigid body we have the following six equations

$$\mu(N_{2,y}+N_{3,y}) = m\ddot{x}_C \tag{1}$$
$$N_{1,y}+N_{2,y}+N_{3,y}+N_{4,y}-mg = 0 \tag{2}$$
$$N_{1,z}+N_{2,z}+N_{3,z}+N_{4,z} = 0 \tag{3}$$
$$\mu(N_{2,y}+N_{3,y})a + b(-N_{1,y}+N_{2,y}+N_{3,y}-N_{4,y}) = 0 \tag{4}$$
$$c(-N_{1,y}-N_{2,y}+N_{3,y}+N_{4,y}) - a(N_{1,z}+N_{2,z}+N_{3,z}+N_{4,z}) = 0 \tag{5}$$
$$b(N_{1,z}-N_{2,z}-N_{3,z}+N_{4,z})) = 0 \tag{6}$$

Since we have 9 unknowns and only 6 equations,, the system is indeterminate and we need to make some assumptions to come up with extra equations or eliminate some unknowns. We make the following assumption

$$N_{1,y} = N_{4,y} = N_{front}, \qquad N_{2,y} = N_{3,y} = N_{rear}, \qquad N_{1,z} = N_{2,z} = N_{3,z} = N_{4,z} = 0$$

This gives us the following equations

$$2\mu N_{front} = m\ddot{x}_C \tag{7}$$
$$2N_{front} + 2N_{rear} - mg = 0 \tag{8}$$
$$\mu(N_{front})a + b(N_{front} - N_{rear}) = 0 \tag{9}$$

where we have 3 equations and 3 unknowns which we can solve. Thus, from (9) we have,

$$N_{front} = \left(\frac{b}{\mu a + b}\right) N_{rear}$$

Using above in (8) we have,

$$\boxed{N_{rear} = \left(\frac{\mu a + b}{\mu a + 2b}\right)\frac{mg}{2}}, \qquad \Rightarrow \boxed{N_{front} = \left(\frac{b}{\mu a + 2b}\right)\frac{mg}{2}}$$

Using this in (7) we have,

$$\ddot{x}_C = \left(\frac{\mu b g}{\mu a + 2b}\right) = K \tag{10}$$

which is the acceleration of the center of mass of the cart. ∎

2) Euler's Disk (spinning/rolling)

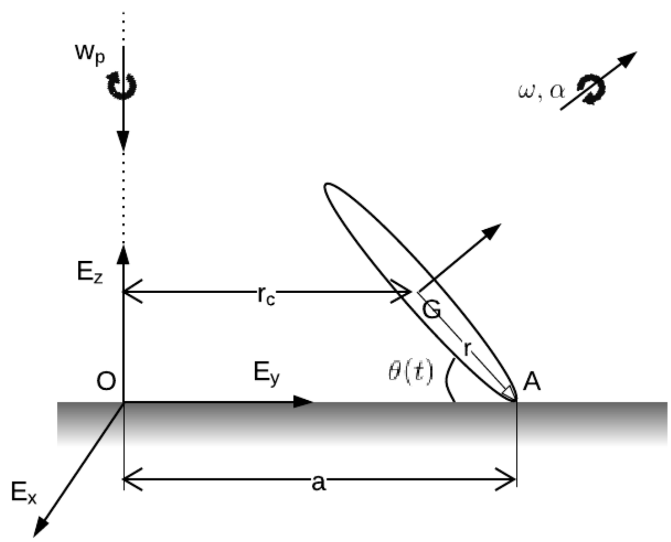
Link to a video depicting the motion of the Euler's disk:

`https://www.youtube.com/watch?v=qdYS8PyOZ7w`

As can be seen from the video, the spinning/rolling action of the Euler's disk is similar to what happens when you spin a coin on a flat surface. The disk spins faster and faster as time goes on before eventually hitting the ground and stopping.
How is it that the angular momentum of the disk keeps increasing with time, seemingly going against our ideas of conservation of angular momentum and energy?

To analyze the dynamics behind the motion of the Euler's disk, we first draw a schematic of the disk as shown on the right, making the following simplifying assumptions:

1. ω_p, the rate of precession, ω_s, the rate of spin, θ, the angle the disk makes with the horizontal, are assumed constant (valid since the rate at which these quantities change with time are small).
2. The disk rolls on the flat surface without slipping.
3. The disk is thin relative to its radius.
4. The disk rolls such that the diameter of the circular path traversed by point A approaches the diameter of the disk itself, as θ becomes small. This implies, we can assume the distance of G from the $\hat{E}_z$ axis, $r_C = a - r\cos\theta = 0$ in the figure to the right.



Now, the angluar velocity of the disk with respect to the ground frame of reference is given by,

$$\vec{\omega} = \omega_s \sin\theta \hat{e}_y + (\omega_s \cos\theta - \omega_p)\hat{E}_z \tag{11}$$

where, $[\hat{e}_x, \hat{e}_y, \hat{e}_z]$ is the rotating frame of reference while $[\hat{E}_x, \hat{E}_y, \hat{E}_z]$ is the fixed global frame of reference. In the above, $\hat{e}_z = \hat{E}_z$. We have the relation between the two frames of reference as,

$$\begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{bmatrix}$$

and also,

$$\frac{d\hat{e}_x}{dt} = \vec{\omega}_p \times \hat{e}_x = \omega_p(-\hat{e}_z \times \hat{e}_x) = -\omega_p \hat{e}_y, \tag{12}$$

$$\frac{d\hat{e}_y}{dt} = \vec{\omega}_p \times \hat{e}_y = \omega_p(-\hat{e}_z \times \hat{e}_y) = \omega_p \hat{e}_x, \tag{13}$$

and $\frac{d\hat{e}_z}{dt} = 0$.

Now, the angular acceleration of the disk with respect to the ground is given by,

$$\begin{aligned} \vec{\alpha} = \frac{d\vec{\omega}}{dt} &= \frac{d[\omega_s \sin\theta \hat{e}_y]}{dt} + \frac{d[(\omega_s \cos\theta - \omega_p)\hat{e}_z]}{dt} \\ &= \omega_s \sin\theta \frac{d\hat{e}_y}{dt} + 0 \end{aligned} \tag{14}$$

since, $\omega_s, \theta, \omega_p$ and $\hat{e}_z = \hat{E}_z$ are assumed to be constant with respect to time. Now, using (3) in (4) we have,

$$\vec{\alpha} = \omega_s \omega_p \sin\theta \hat{e}_x \tag{15}$$

Now consider the point A on the disk. The linear velocity of A is given by,

$$\vec{v}_A = \vec{v}_G + \omega_s(\sin\theta\hat{e}_y + \cos\theta\hat{e}_z) \times r(\cos\theta\hat{e}_y - \sin\theta\hat{e}_z)$$

and,

$$\vec{v}_G = -\omega_p \hat{e}_z \times a\hat{e}_y = a\omega_p \hat{e}_x$$

Thus, we have

$$\vec{v}_A = (a\omega_p - r\omega_s)\hat{e}_x$$

For pure rolling, the point A has zero velocity. This implies,

$$a\omega_p = r\omega_s$$

Also, $a = r\cos\theta$ (from assumption 4) which gives us the relation between the rate of precession and the rate of spin as,

$$\omega_p = \frac{\omega_s}{\cos\theta} \tag{16}$$

Now, for the condition that $r_c = 0$, the point G (the center of mass of the disk) is stationary, implying that the acceleration of the point G is zero. Thus, the normal force acting at point A, $\vec{N} = N\hat{e}_z$ balances the gravitational force, $-mg\hat{e}_z$ giving,

$$N = mg$$

Also, the torque due to this normal force results in ,

$$\tau = r(\cos\theta\hat{e}_y - \sin\theta\hat{e}_z) \times mg(\hat{e}_z) = I_{Gx}\vec{\alpha}$$

where, $I_{Gx} = mr^2/4$ for the thin disk and $\vec{\alpha}$ is given by (5). Thus, we have,

$$mgr\cos\theta\hat{e}_x = \frac{mr^2}{4}\omega_s\omega_p\sin\theta\hat{e}_x \tag{17}$$

Using (6) in (7) we have,

$$\omega_p^2 = \frac{4g}{r\sin\theta}$$

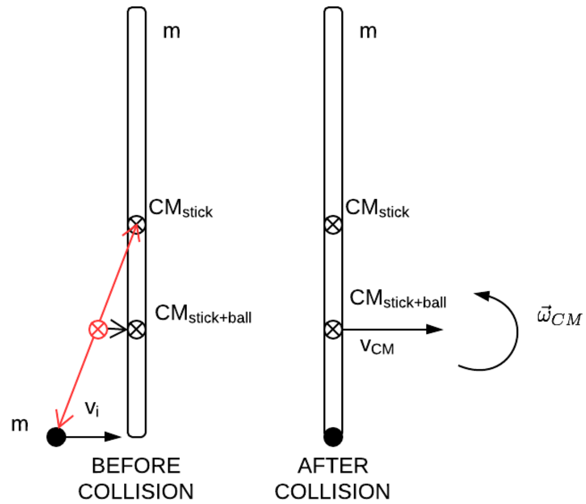
$$\Rightarrow \boxed{\omega_p = \sqrt{\frac{4g}{r\sin\theta}}}$$

This implies that as $\theta \to 0$ we have that $\omega_p \to \infty$ due to the $\sin\theta$ term in the denominator. This is consistent with the rapid increase in the angular velocity we see in a real Euler's disk as θ becomes small. In reality, there is a bit of slipping (friction) at point A (along with rolling friction) which results in the disk losing energy. This results in an equal loss in potential energy (conservation of energy) leading to a loss in height of the center of mass, G (leading to a decrease in θ). Thus, ω_p keeps increasing as θ keeps decreasing upto the point where the disk hits the ground and is suddenly brought to a stop.

Although we have neglected friction, angular momentum is still **not** conserved about the center of mass G due to the net external torque acting about G due to the normal reaction force from the ground at point A. ■

3) Stick and ball collision

Given a long narrow uniform stick of length l and mass m which lies motionless on a frictionless surface (the figure on the right is the top view). The center of mass of the stick is at the geometric center of the stick and its moment of inertia about CM_{stick} is $I_{CM} = ml^2/12$. A ball has the same mass m and slides without spinning on the on the surface with a velocity $\vec{v}_i$ toward the stick, hits one end of the stick, and attaches onto it. Assume that the radius of the ball is much smaller than the length of the stick and so that the moment of inertia of the ball about its center is negligible compared to I_{CM}.



(a) How far from the midpoint of the stick is the center of mass of the stick-ball combination after the collision?

(b) What is the linear velocity $\vec{v}_{CM,f}$ of the center of mass of the stick-ball combination after the collision?

(c) Is mechanical energy conserved during the collision? Explain your reasoning.

(d) What is the angular velocity $\vec{\omega}_{CM,f}$ of the stick-ball combination after the collision?

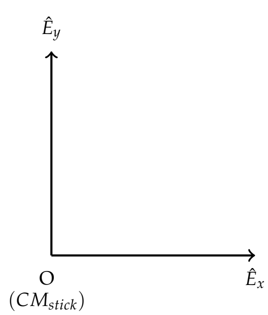
(e) How far does the stick's center of mass move during one rotation of the stick?

Solution

(a) Note that we are considering the configurations before and after collisions as ones *immediately* before and after collision. This implies the location of the ball is the same in both the configurations.

The position vector of the center of mass of a system of particles is given by,

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$



In this case, let us first define the coordinate system as the Cartesian coordinate system with the origin at CM_{stick} shown in the figure. Thus, the distance of the center of mass of the stick-ball combination from CM_{stick} ($d_i = CM_i - CM_{stick}$) is given by,

$$\boxed{d_{cm} = \frac{m_{stick} d_{stick} + m_{ball} d_{ball}}{m_{stick} + m_{ball}} = \frac{0 + ml/2}{m + m} = \frac{l}{4}} \tag{18}$$

■

(b) Since there is no external force acting on the *system* (comprising of the ball and the stick) we have by Newton's second law,

$$\vec{F}_{external} = \frac{d\vec{p}}{dt} = 0, \qquad \Rightarrow \vec{p} = constant \tag{19}$$

which implies that the linear momentum of the *system* is conserved before and after collision. The initial linear momentum is just due to the ball since the stick is stationary. It is given by,

$$\vec{p}_i = m\vec{v}_i = mv_i\hat{E}_x \tag{20}$$

And the linear momentum of the *system* after the collision is given by,

$$\vec{p}_f = (m+m)\vec{v}_{cm,f} = 2mv_{cm,f}\hat{E}_x \tag{21}$$

Where we know that the direction of the linear velocity of the center of mass of the *system* is only along $\hat{E}_x$. This is because $CM_{stick+ball}$ was initially moving along $\hat{E}_x$ as shown in red in the figure, and there is no external force to change its direction. Thus, after collision it continues to move in the same direction.

Thus, using (21) and (20) in (19) we have,

$$mv_i = 2mv_{cm,f}, \qquad \Rightarrow v_{cm,f} = \frac{v_i}{2}$$

Thus, the linear velocity of the center of mass of stick-ball combination after collision is given by

$$\boxed{\vec{v_{cm,f}} = \frac{v_i}{2}\hat{E}_x}$$

■

(c) Given that the ball gets attached to the stick after collision. This implies that the velocity of the ball and that of the stick at that point are the same. Now, the coefficient of resititution for the collision is defined as,

$$e = \frac{\text{relative velocity of departure after collision}}{\text{relative velocity of approach before collision}} = \frac{\{v_{stick,A}\}_f - \{v_{ball}\}_f}{\{v_{ball}\}_i - \{v_{stick,A}\}_i} = \frac{0}{v_i} = 0$$

Thus, the collision is completely inelastic, which means that some portion of the mechanical energy is lost. Thus, mechanical energy is **NOT** conserved. We will see this explicitly at the part (d) after computing the angular velocity of the stick-ball combination after collision. ■

(d) Similar to the arguments for the linear momentum conservation, the angular momentum is also conserved since there is no external torque acting on the *system* (consisting of the ball and the stick). That is,

$$\vec{\tau}_{external} = \frac{d\vec{L}}{dt} = 0, \qquad \Rightarrow \vec{L} = constant \tag{22}$$

Consider the angular momentums about $CM_{stick+ball}$. The initial angular momentum of the system is only due to the ball and is given by,

$$\vec{L}_i = \vec{r}_{ball} \times \vec{p}_{ball} + \vec{r}_{stick} \times \vec{p}_{stick} = \frac{l}{4}(-\hat{E}_y) \times mv_i(\hat{E}_x) + 0 = mv_i\frac{l}{4}\hat{E}_z \tag{23}$$

and similarly, the angular momentum of the *system* about $CM_{stick+ball}$ after collision is given by,

$$\vec{L}_f = I_{cm,sys}\vec{\omega_{cm,f}} \tag{24}$$

where $I_{cm,sys}$ is the moment of inertia of the stick+ball about the axis passing through $CM_{stick+ball}$ and parallel to $\hat{E}_z$. Now, by the parallel axis theorem we have for the stick,

$$I_{cm,stick} = \frac{ml^2}{12} + md_{cm}^2 = \frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2 = \frac{7}{48}ml^2$$

and, for the ball about $CM_{stick+ball}$ we have,

$$I_{cm,ball} = m\left(\frac{l}{4}\right)^2 = \frac{1}{16}ml^2$$

Thus, the moment of inertia of the combined system is given by,

$$I_{cm,sys} = I_{cm,stick} + I_{cm,ball} = \frac{7}{48}ml^2 + \frac{1}{16}ml^2 = \frac{5}{24}ml^2 \tag{25}$$

Thus, using (25) in (24) and assuming the direction of $\vec{\omega}_{cm,f}$ as $\hat{E}_z$, we have

$$\vec{L}_f = \frac{5}{24}ml^2\omega_{cm,f}\hat{E}_z \tag{26}$$

Thus, using (26) and (23) in (22) we have,

$$mv_i\frac{l}{4} = \frac{5}{24}ml^2\omega_{cm,f} \quad \Rightarrow \omega_{cm,f} = \frac{6v_i}{5l}$$

Thus, the angular velocity of the stick-ball system is given by,

$$\boxed{\vec{\omega}_{cm,f} = \frac{6v_i}{5l}\hat{E}_z}$$

We can now see if the argument that the mechanical energy of the system being *NOT* conserved is actually true. The energy of the system before (T_i) and after collision (T_f) are given by,

$$\begin{aligned}
T_i &= \frac{1}{2}mv_i^2, \\
T_f &= \frac{1}{2}(m+m)v_{cm,f}^2 + \frac{1}{2}I_{cm,sys}\omega_{cm,f}^2 \\
&= \frac{1}{2}(2m)\left(\frac{v_i}{2}\right)^2 + \frac{1}{2}\left(\frac{5}{24}ml^2\right)\left(\frac{6v_i}{5l}\right)^2 \\
&= \frac{mv_i^2}{4} + \frac{36\times 5}{48\times 25}mv_i^2 \\
&= \frac{2}{5}mv_i^2
\end{aligned}$$

Thus, we see that $T_i \neq T_f$ and hence, proved that the mechanical energy of the system is not conserved. ■

(e) The time taken for once complete rotation of the stick+ball system after collision is given by,

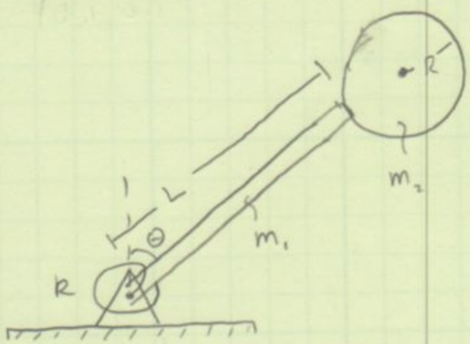
$$T = \frac{2\pi}{\omega_{cm,f}} = \frac{10\pi}{6}\frac{l}{v_i}$$

Thus, the distance moved by the center of mass of the stick-ball system in this time is given by,

$$\boxed{x_{cm} = v_{cm,f}T = \frac{v_i}{2}\left(\frac{10\pi}{6}\frac{l}{v_i}\right) = \frac{5\pi}{6}l}$$

■

Hibbeler 18.9 (the O'Reilly way)



• Given $m_1, m_2, k, L, R, \theta_1, \theta_2, g$,

Determine $\dot{\theta}$ when $\theta = \theta_2$

$m_1 = 6\,kg$, $m_2 = 9\,kg$, $k = 20\ \frac{N\cdot m}{rad}$, $L = 0.45\,m$

$R = 0.075\,m$, $\theta_1 = 0°$, $\theta_2 = 90°$, $g = 9.81\,m/sec^2$

Let's use work energy, which states

$$\frac{dT}{dt} = \sum \underset{\sim}{F} \cdot \underset{\sim}{v}$$

Or, the rate of change of the systems kinetic energy is equal to the mechanical power of the resultant force (+ moments)

$$\int_{T_1}^{T_2} dT = \sum \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}\, dt$$

$$T_2 - T_1 = \sum \int_{r_1}^{r_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

• Thus, the change in potential energy is equal to the work done to the system by moments and forces.

The kinetic energy for a system of bodies is

$$T = \sum_{n=1}^{N} \left(\frac{1}{2} m_n \dot{\underset{\sim}{X}}_n \cdot \dot{\underset{\sim}{X}}_n + \frac{1}{2} \underset{\sim}{H}_n \cdot \underset{\sim}{\omega}_n \right)$$

Where $\dot{X}_n$ is the velocity of the n^{th} center of mass and $\underset{\sim}{H}_n$ is the angular momentum of the n^{th} mass, or

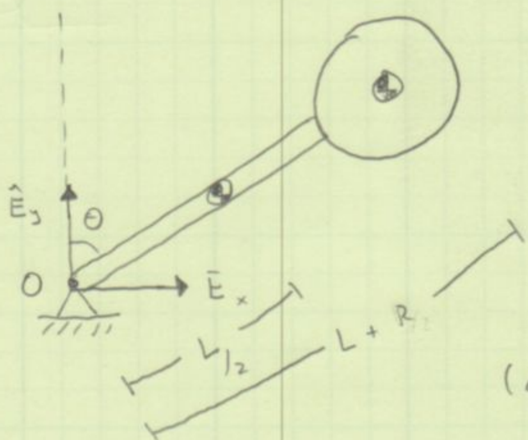
$$\underset{\sim}{H}_n = I_{G,n}\, \underset{\sim}{\omega}_n = I_{G,n}\, \dot{\underset{\sim}{\theta}}$$

Hibbeler 18.9 (the O'Reilly way)

Since the body is initially at rest, it follows that

$T_1 = 0$

To determine T_2, we draw a position vector to each center of mass,



(define $L + R \stackrel{\Delta}{=} B$ for tidyness)

$B = 0.525$ m

$$\underset{\sim}{\bar{X}}_1 = \frac{L}{2}\sin\theta\,\hat{E}_x + \frac{L}{2}\cos\theta\,\hat{E}_y$$

$$\underset{\sim}{\bar{X}}_2 = B\sin\theta\,\hat{E}_x + B\cos\theta\,\hat{E}_y$$

(we could have used a rotating coordinate frame if we had wanted)

Next, we differentiate the position vectors to give us the velocity of the center of mass,

$$\frac{d}{dt}(\underset{\sim}{\bar{X}}_1) = \dot{\underset{\sim}{\bar{X}}}_1 = \frac{L}{2}\dot{\theta}\cos\theta\,\hat{E}_x - \frac{L}{2}\dot{\theta}\sin\theta\,\hat{E}_y$$

$$\frac{d}{dt}(\underset{\sim}{\bar{X}}_2) = \dot{\underset{\sim}{\bar{X}}}_2 = B\dot{\theta}\cos\theta\,\hat{E}_x - B\dot{\theta}\sin\theta\,\hat{E}_y$$

Hibbeler 18.9 (the O'Reilly way)

Then,

$$\dot{\underset{\sim}{X}}_1 \cdot \dot{\underset{\sim}{X}}_1 = \frac{L^2}{4}\dot{\theta}^2\cos^2\theta + \frac{L^2}{4}\dot{\theta}^2\sin^2\theta = \frac{L^2}{4}\dot{\theta}^2$$

$$\dot{\underset{\sim}{X}}_2 \cdot \dot{\underset{\sim}{X}}_2 = B^2\dot{\theta}^2\cos^2\theta + B^2\dot{\theta}^2\sin^2\theta = B^2\dot{\theta}^2$$

Thus, the kinetic energy is

$$T = \frac{1}{8}m_1 L^2\dot{\theta}^2 + \frac{1}{2}m_2 B^2\dot{\theta}^2 + \frac{1}{2}I_{G,1}\dot{\theta}^2 + \frac{1}{2}I_{G,2}\dot{\theta}^2$$

where $I_{G,1} = \frac{1}{12}m_1 L^2$ (MOI of slender rod about centroid)

$I_{G,2} = \frac{1}{2}m_2 R^2$ (MOI of disc about centroid)

Thus, numeric evaluation yields

$$T = 1.455\,\dot{\theta}^2 \text{ J}$$

- Keep in mind, this could have been simplified using the idea of rotation about a fixed point, as $\dot{X}_1$ + $\dot{\theta}$ are directly related. You would also need to use the moments of inertia about fixed point 'O'.

Next, we wish to determine the work done to the system. Work is done by gravity (constant force) and the torsional spring (variable force). We will evaluate each term independently.

Hibbeler 18.9 (the O'Reilly way)

First, we'll make a slight notation shift and allow

$$\bar{\underset{\sim}{X}}_n = \underset{\sim}{r}_n \text{, so}$$

$$\sum \int_{\underset{\sim}{r}_1}^{\underset{\sim}{r}_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \sum \int_{(\bar{\underset{\sim}{X}}_n)_1}^{(\bar{\underset{\sim}{X}}_n)_2} \underset{\sim}{F} \cdot d\underset{\sim}{X} = U_N$$

Lets look at the work done by gravity to the first mass.

$$\bar{\underset{\sim}{X}}_1 = \frac{L}{2}\sin\theta \, \hat{E}_x + \frac{L}{2}\cos\theta \, \hat{E}_y$$

Applying the principal of virtual work,

$$\delta \underset{\sim}{X}_1 = \left[\frac{L}{2}\cos\theta \, \hat{E}_x - \frac{L}{2}\sin\theta \, \hat{E}_y\right]\delta\theta$$

Gravity only acts in the Y-direction, so

$$\underset{\sim}{F}_1 = -m_1 g \, \hat{E}_y$$

Thus,

$$\underset{\sim}{F}_1 \cdot d\underset{\sim}{X}_1 = m_1 g \frac{L}{2}\sin\theta \, \delta\theta$$

and

$$U_1 = \int_{(\underset{\sim}{X}_1)_1}^{(\underset{\sim}{X}_1)_2} \underset{\sim}{F}_1 \cdot d\underset{\sim}{X}_1 = m_1 g \frac{L}{2} \int_{\theta_1}^{\theta_2} \sin\theta \, \delta\theta = -m_1 g \frac{L}{2}\left[\cos\theta_2 - \cos\theta_1\right|_{\theta_1 = 0°}^{\theta_2 = 90°}$$

$$U_1 = m_1 g \frac{L}{2} \qquad \therefore U_1 = 13.24 \text{ J}$$

The procedure for the work done by gravity to the second mass is nearly identical, except with a different radius

Hibbeler 18.9 (the O'Reilly way)

$$\bar{\underset{\sim}{X}}_2 = B\sin\theta\,\hat{E}_x + B\cos\theta\,\hat{E}_y$$

$$\delta \underset{\sim}{X}_2 = [B\cos\theta\,\hat{E}_x - B\sin\theta\,\hat{E}_y]\,\delta\theta$$

$$\underset{\sim}{F}_2 = -m_2 g\,\hat{E}_y$$

$$\underset{\sim}{F}_2 \cdot \delta\underset{\sim}{X}_2 = B m_2 g \sin\theta\,\delta\theta$$

$$U_2 = \int_{(\bar{X}_2)_1}^{(\bar{X}_2)_2} \underset{\sim}{F}_2 \cdot \delta\underset{\sim}{X}_2 = B m_2 g \int_{\theta_1}^{\theta_2} \sin\theta\,\delta\theta = -m_2 g B[\cos\theta_2 - \cos\theta_1]\Big|_{\theta_1 = 0}^{\theta_2 = 90°}$$

$$U_2 = m_2 g B \quad \therefore \; U_2 = 46.35\text{ J}$$

Lastly, the work done by the torsional spring.

Let

$$\bar{\underset{\sim}{X}}_3 = \theta\,\hat{E}_z, \quad \delta\bar{\underset{\sim}{X}}_3 = \delta\theta\,\hat{E}_z$$

Then, $\underset{\sim}{F}_3 = -k\theta\,\hat{E}_z$

$$\underset{\sim}{F}_3 \cdot \delta\bar{\underset{\sim}{X}}_3 = -k\theta\,\delta\theta$$

$$U_3 = \int_{(X_3)_1}^{(X_3)_2} \underset{\sim}{F}_3 \cdot \delta\bar{\underset{\sim}{X}}_3 = -k\int_{\theta_1}^{\theta_2} \theta\,\delta\theta = -\frac{k}{2}[\theta^2\Big|_{\theta_1=0}^{\theta_2=\pi/2} = -\frac{k\pi^2}{8}$$

$$U_3 = -24.67\text{ J}$$

Finally,

$$U_1 + U_2 + U_3 = T_2$$

$$34.92\text{ J} = 1.455\,\dot{\theta}^2$$

$$\boxed{\dot{\theta} = 4.90\ \text{rad/sec}}$$