11. Structures: Complex Stresses and Deflections

Engineering structures may take many forms, from the simple shapes of square cross section beams to the complex and intricate shapes of trusses. Regardless of the shape another important aspect of structures is that often the stresses, strains, and deflections of the structure do not lend themselves to simple and straight-forward analyses of simple components such as those used for materials testing (e.g., the uniaxially-loaded and uniformly-stressed tensile specimen). Further complicating the analyses of engineering structures is the need to apply failure criteria to evaluate the probable success (or non success) of the design.

Failure Criteria

Engineering failure can be broadly defined as the "inability to perform the intended function." An obvious failure is a broken part (unless of course the intended function is to fail as in the case of shear pins or explosive bolts!) which is known as fracture. However, excessive elastic or plastic deformation without fracture can also constitute a failure. In addition, a component with too much or not enough "give" such as with too compliant or too stiff of a spring-like component can be a failure. A cracked component such as in a pressure vessel would constitute a failure if a leak occurred. Thus, failure criteria can be based on stress, strain, deflection, crack length, time or cycles, or any other engineering parameter we choose to apply.

The most common failure criteria are stress-based. The basic premise is that failure will occur in the component or structure when the combined stress state is equal that which caused failure in the same material subjected to a uniaxial tensile test. Two primary types of stress-based failure criteria are used: yield (for ductile materials with %el>5) and fracture (for brittle materials with %el<5). In both cases, one can consider failure to occur at the onset of non linearity in the tensile stress strain curve (the yield point in ductile materials and ultimate tensile strength in brittle materials).

Fracture Criterion: The simplest failure criterion is that failure is expected when the greatest principal stress reaches the uniaxial tensile strength of the material. Thus, the Maximum Normal Stress fracture criterion (a.k.a., Rankine) can be specified as:

$$\text{Fractures if } \mathrm{MAX}\left[|\sigma_1|, |\sigma_2|, |\sigma_3|\right] \geq S_{UTS} \qquad (11.1)$$

where the function MAX indicate the greater of the absolute values of the principal normal stresses. Note that it is assumed that the ultimate strength of the material is the same in compression or tension.

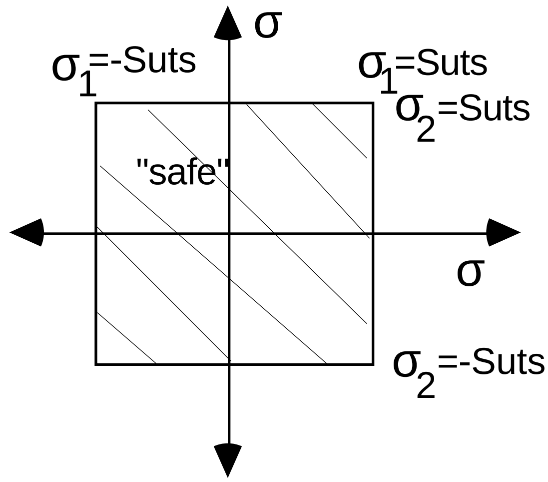


Figure 11.1 Failure envelope for maximum normal stress criterion

A factor of safety can be defined based on Eq. 11.1 such that:

$$\text{Fractures if FS} \le 1 \text{ where FS}= \frac{S_{UTS}}{\text{MAX}\left[|\sigma_1|,|\sigma_2|,|\sigma_3|\right]} \tag{11.2}$$

This fracture criterion can be represented in a plane stress state (σ_z=0) where σ_2 is the ordinate and σ_1 is the abscissa. As shown in Fig. 11.1, any combination of σ_1 and σ_2 that falls within the square box (i.e., FS=1 for Eq. 11.2 where $\pm S_{UTS}=\sigma_1$ or $\pm S_{UTS}=\sigma_2$) is "safe" and the perimeter is fracture.

Yield criteria: There are two relatively well-accepted yield criteria: Maximum Shear Stress criterion (a.k.a., Tresca) and Octahedral Shear Stress criterion (a.k.a., Distortional Energy or Von Mises). Each is discussed as follows.

The simplest yield criterion is that yield failure is expected when the greatest shear stress reaches the shear strength of the material. Thus, the maximum shear stress yield criterion can be specified as:

$$\text{Yields if MAX}\left[\tau_{12}=\left|\frac{(\sigma_1-\sigma_2)}{2}\right|,\tau_{13}=\left|\frac{(\sigma_1-\sigma_3)}{2}\right|,\tau_{23}=\left|\frac{(\sigma_2-\sigma_3)}{2}\right|\right] \ge \tau_o = \frac{\sigma_o}{2} \tag{11.3}$$

where the function MAX indicates the greater of the absolute values of the principal shear stresses.

A factor of safety can be defined based on Eq. 11.3 such that:

Yields if FS $\le$ 1

$$\text{where FS}= \frac{\tau_o=\sigma_o/2}{\text{MAX}\left[\tau_{12}=\left|\frac{(\sigma_1-\sigma_2)}{2}\right|,\tau_{13}=\left|\frac{(\sigma_1-\sigma_3)}{2}\right|,\tau_{23}=\left|\frac{(\sigma_2-\sigma_3)}{2}\right|\right]} \tag{11.4}$$

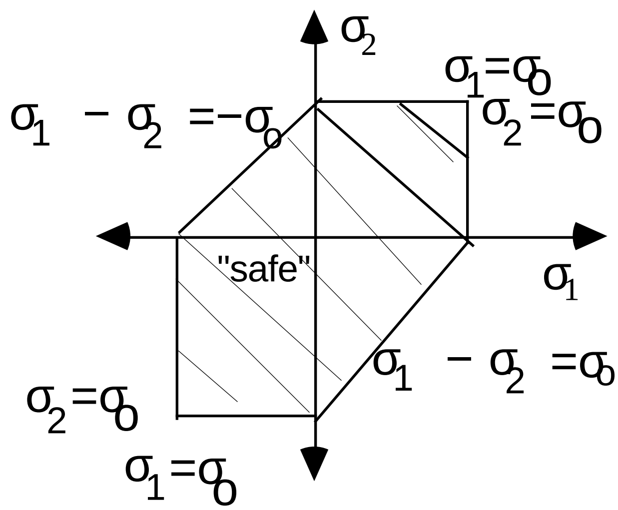


Figure 11.2 Failure envelope for maximum stress criterion

This yield criterion can be represented in a plane stress state (σ_z =0) where σ_2 is the ordinate and σ_1 is the abscissa. As shown in Fig. 11.2, any combination of σ_1 and σ_2 that plots within the parallelogram (i.e., FS=1 for Eq. 11.4 where $\pm\sigma_o=(\sigma_1-\sigma_2)$, $\pm\sigma_o=\sigma_1$ or $\pm\sigma_o=\sigma_2$) is "safe" and the perimeter is yielding.

A more complicated yield criterion is that yield failure is expected when the octahedral shear stress,τ_h, reaches the octahedral shear stress at yield of the material, τ_{ho}. Thus, the octahedral shear stress yield criterion can be specified as:

$$\text{Yields if } \tau_h \geq \tau_{ho}$$
$$\text{where } \tau_h = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \tag{11.5}$$

and

$$\tau_{ho} = \frac{\sqrt{2}}{3}\sigma_o \tag{11.6}$$

when the stress state of a uniaxial tensile test at yielding ($\sigma_1=\sigma_o$, $\sigma_2=\sigma_3=0$) are substituted into the relation for τ_h given in Eq. 11.5. If Eq 11. 5 and 11.6 are set equal to each other the yield criterion can be expressed in terms of normal stresses:

$$\text{Yields if } \tau_h \geq \tau_{ho} \qquad \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \geq \frac{\sqrt{2}}{3}\sigma_o \tag{11.7}$$

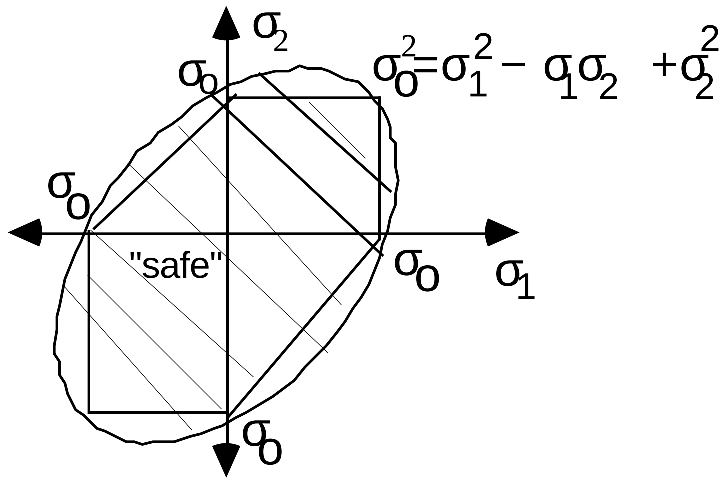


Figure 11.3 Failure envelope for maximum stress criterion

A factor of safety can be defined based on Eq. 11.7 such that:

Yields if FS ≤ 1

$$\text{where FS}=\frac{\sigma_o}{\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}}=\frac{\sigma_o}{\bar{\sigma}_H} \qquad (11.8)$$

where $\bar{\sigma}_H$ is the effective stress based on the octahedral shear stress criterion.

This yield criterion can be represented in a plane stress state (σ_z =0) where σ_2 is the ordinate and σ_1 is the abscissa. As shown in Fig. 11.3, any combination of σ_1 and σ_2 that plots within the ellipse (i.e., FS=1 for $\sigma_o^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$ is "safe" and the perimeter is yielding.

The usefulness of the three failure criteria presented here is shown in Fig 11.4 for the failure envelopes for the plane stress case where σ_1 and σ_2 are normalized to S_{UTS} or σ_o. Note for the brittle material (cast iron) that the actual failure points follow the maximum normal stress criterion envelope (i.e., FS=1) and for the ductile materials (steels and aluminums) that the actual failure points fall between the maximum shear stress and octahedral shear stress criteria envelopes (i.e., FS=1). Since the maximum difference between the two yield criteria is about 15%, it is often advisable to err on the side of conservatism and use the simpler maximum shear stress criterion for ductile materials.

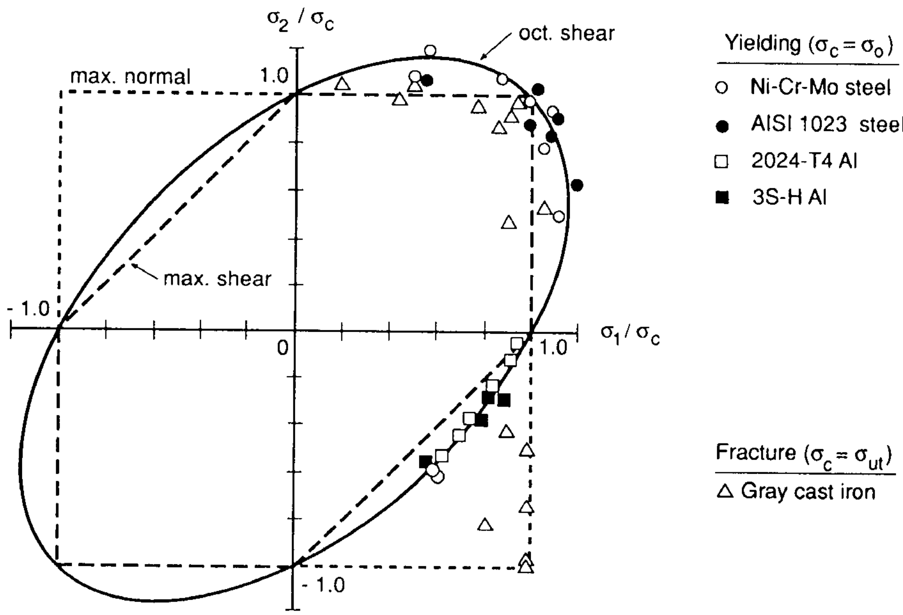


Figure 11.4 Failure criteria and failure points plotted on normalized plane stress coordinates

<u>Combined Stresses</u>

The previous discussion of failure criteria was based on two premises: 1) the uniaxial tensile behavior of the material was known (i.e., S_{UTS} or σ_o.) and 2) the principal stresses based on all the coordinate stresses was known.

When determining the mechanical properties and performance of a material, such as its yield strength, it is desirable to choose a fundamental test that will give the required property in the most direct manner. Thus, for yield strength, a simple one-dimensional tensile test specimen is usually used. Figure 11.5 illustrates the fundamental tensile test and shows a free body diagram of an infinitesimal element with the one-dimensional stress σ_z acting on it

On the other hand, in a realistic situation, the engineer is usually faced with a two- or three-dimension load condition in which, at any point, P, the loaded member may be subject to a combination of tension, compression and shear stresses as idealized in Fig. 11.6.

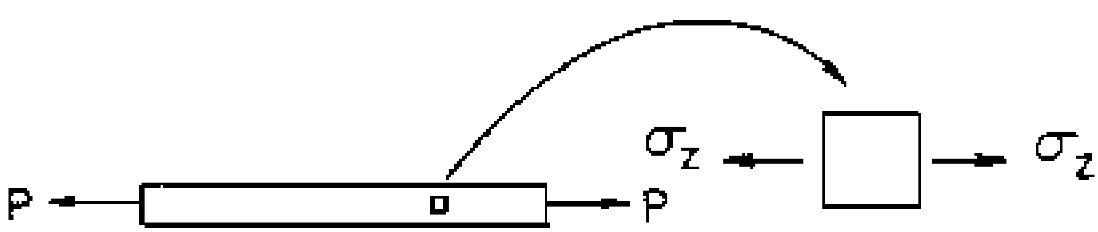


Figure 11.5 Illustration of the uniaxial tensile test

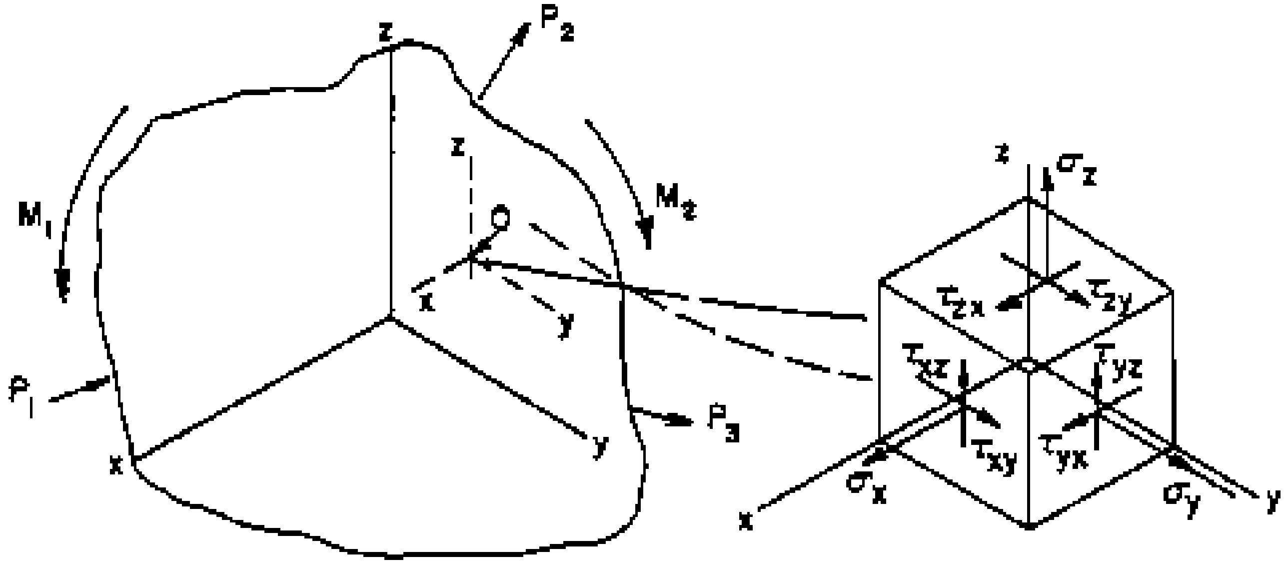


Figure 11.6. General three-dimensional stress state

Figure 11.6 shows an arbitrary body loaded with forces P and moments M. At a point O, shown enlarged at the right, an infinitesimal three-dimensional element can be acted upon by normal stresses σ_x, σ_y and σ_z acting in the x, y and z directions, as shown. Often, the stresses in the z directional are zero, or much smaller than σ_x or σ_y. This is a condition called plane stress, and the analysis is simplified.

Thus, in order to determine the margin of safety of a loaded structure, it is necessary to relate the two- or three-dimensional stress state that usually occurs, with the fundamental strength, like tensile yield strength, that is obtained in the laboratory. This relationship is defined through the previously-discussed failure criteria of which there are a number in addition to those already discussed.

Prerequisite to applying a failure criterion, is to deduce from the general two- or three-dimensional element in which both shear and normal stresses are present, the principal stresses σ_1 and σ_2 and the maximum shear stresses τ_{max}. These stresses can be determined either analytically or experimentally.

The analytical calculation of principal stress and maximum shear stress involves the superposition of normal and shear stress to determine the total stress acting at a critical point. Thus, normal stresses are computed from P/A for simple tension and Mc/I for bending. Shear stresses are computed from Tr/J for torsion and VQ/It for direct shear. For thin-walled, pressurized cylindrical pressure vessels the hoop stress is given by pr/t and the axial stress by pr/2t.

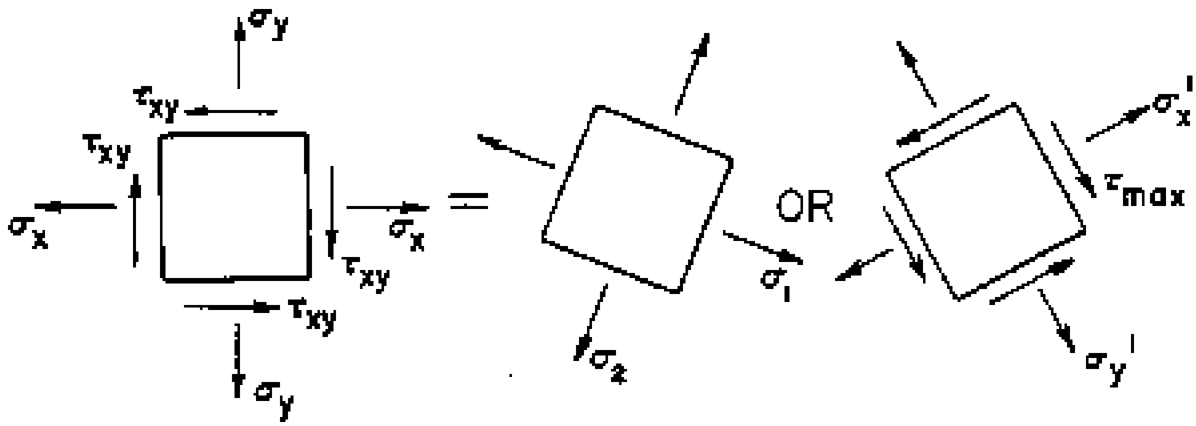


Figure 11.7 A general two-dimensional stress state and stresses resulting from element rotation.

Since the final stress state is independent of the order in which the loads are applied, the total stress existing at a point with a combined loading consisting, perhaps, of tension, torsion, pressure and shear can be found by simply adding normal stresses together and shear stresses together taking proper account of their sign. Finally, after considering the stresses caused by each load, a two-dimensional element at the point considered may appear as shown in Figure 11.7.

Then an application of Mohr's circle of stress will give the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max}. Stress is the quantity that causes failure, yet we realize from earlier work, that one cannot measure stress directly. This is because stress is related to the force in a part, which, except in very simple cases, is not easily measured. On the other hand, strain is easily measured and these values can easily be converted to stress values.

Calculation of principal strains can be performed as follows. If we consider only a two-dimensional object, the unknown strains are the normal or elongation strains ε_x and ε_y and the shear strain γ_{xy}. Since electrical resistance strain gages can measure only normal strains (ε), not shear strains (γ), we need to measure three normal strains at a point to determine the three strains ε_x, ε_y, and γ_{xy} at a point. This follows directly from the equations that give strain in a direction "a" (ε_a), oriented at an angle θ from the x axis, as shown in Figure 11.8, when strains along the x and y axes, ε_x and ε_y, and the shear strain γ_{xy} are known. Thus:

$$\varepsilon_a = \frac{1}{2}\left(\varepsilon_x + \varepsilon_y\right) + \frac{1}{2}\left(\varepsilon_x - \varepsilon_y\right)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \qquad (11.9)$$

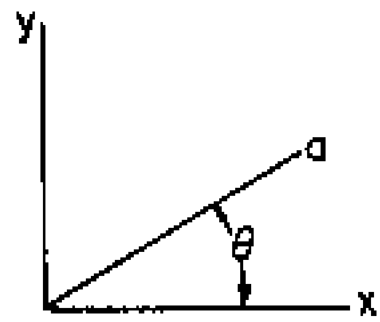


Figure 11.8 Orientation of "a" direction and the x-y axes

If we measure three strains, ε_{a1}, ε_{a2}, ε_{a3}, at three different angles $\theta_1, \theta_2, \theta_3$, we can substitute the values in the above equations and obtain three equations to solve for ε_x, ε_y, and γ_{xy} . Knowing ε_x, ε_y, and γ_{xy}, Mohr's circle of strain can then be used to find ε_1, ε_2, and γ_{max}.

A special type of electrical resistance strain gage called a rosette is available for measuring the three normal strains at a point. These rosettes are simply three strain gages, mounted one directly on top of the other, or near each other, and oriented at precise angular relationship with respect to each other. Several types of rosettes are available, the most common being a rectangular rosette with 45° between gages and a delta rosette with 60° between gages.

Although the analytical method for calculating the principal strains has already been described, a graphical method for accomplishing the same result also exists. The graphical method has the inherent advantages of graphical techniques with the added advantage of directly producing the Mohr's circle of strain, i.e., ε_1, ε_2 and γ_{max} are given directly. This method is known as Murphy's method. This method is described as follows and is illustrated in Fig. 11.9 for a rectangular rosette.

1. Assume we are using a rosette with strain gages oriented along lines a, b, and c oriented at angles α and β apart, in this case, $\alpha = \beta = 45°$. It is desired to find the maximum strains ε_1 and ε_2.

2. Along an arbitrary horizontal axis x - x, lay off three vertical lines a, b, and c corresponding to the three measured strains ε_a. ε_b, ε_c measured from the y axis. The y - y axis will be the $\gamma/2$ shear strain axis.

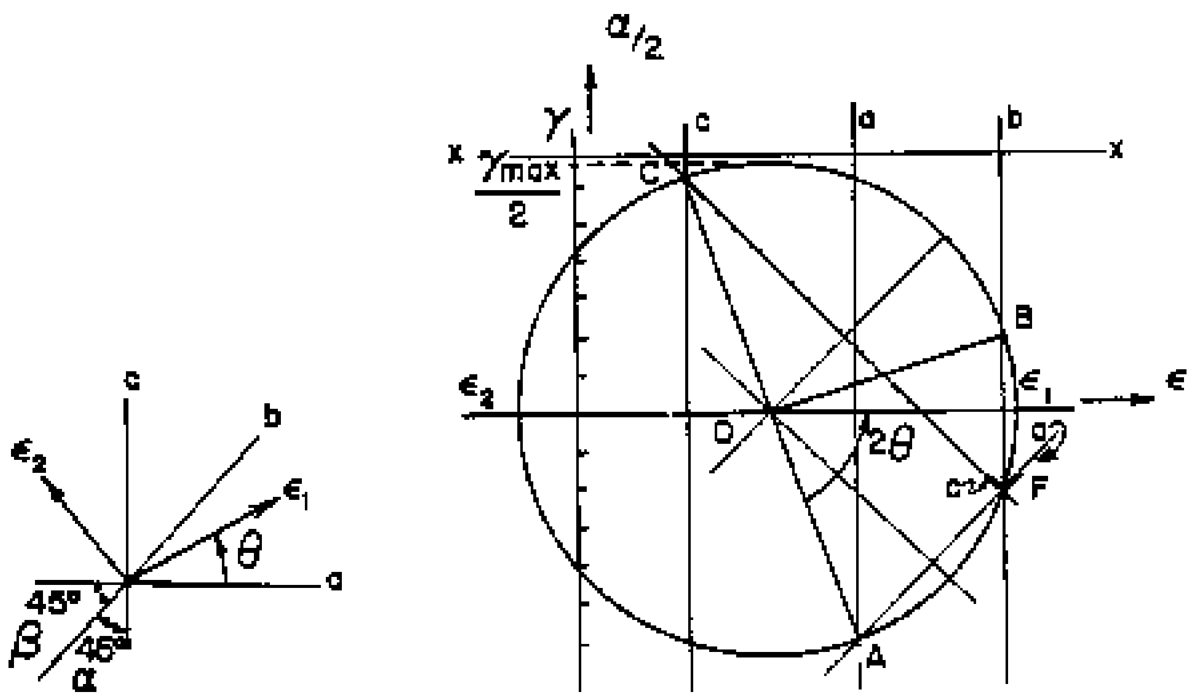


Figure 11.9 Mohr's strain circle as determined graphically from strain valves ε_a. ε_b, and ε_c.

3. Let the b gage direction lie between the a and c directions as shown. Then from a convenient point F on line b, lay off lines whose directions correspond to the direction of the gauging lines of the rosette, maintaining the same directional sense as in the rosette. These lines intersect the lines a and c at points A and C.

4. Erect perpendicular bisectors to line FA and FC, to intersect at O.

5. Draw a circle with center at O and passing through points A, F, and C.

6. From points C, B and A, draw radii to O. Draw the strain axis ε horizontal through O. These radii will have the same angular orientation sense as the corresponding gauging lines of the rosette; the angle between the radii will be twice the actual angle between the gages.

7. The point A, B, and C on the circle give the values of ε and $\gamma/2$ for the three gages.

8. Values of principal strains are determined by the intersection of the circle and the ε axis. The angular orientation of ε_1 from gage A is shown as 2θ.

Thus this simple graphical technique results in a Mohr's circle of strain. Strain values at any angular orientation can be found. Once the principal strains are found the principal stresses follow directly from the Hooke's relations, considering Poisson's effect:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu\sigma_2}{E} \tag{11.10}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\nu\sigma_1}{E} \tag{11.11}$$

or more conveniently, the inverse of these

$$\sigma_1 = \frac{(\varepsilon_1 + \nu\varepsilon_2)E}{(1-\nu^2)} \tag{11.12}$$

$$\sigma_2 = \frac{(\varepsilon_2 + \nu\varepsilon_1)E}{(1-\nu^2)} \tag{11.13}$$

where E is the elastic modulus and ν is Poisson's ratio.

The shear stress-strain relation is completely independent of the normal stress-strain relation and is given by

$$\tau = G\gamma \tag{11.14}$$

where G is the shear modulus of the material.

Types of Engineering Structures

One type of engineering structure is one which is composed of a few simple elements but subjected to a complex loading condition as shown in Fig. 11.10. In this figure the loading condition involves a torque and bending moment and possibly an internal pressure. The stresses due to these loading conditions can be calculated and appropriately superposed before performing the transformations to determine the principal stresses.

Another type of engineering structure is one which is composed of many similarly loaded elements subjected to either a relatively simple or slightly more complex loading condition. Trusses (see Fig. 11.11) are an example of one of the major types of engineering structures, providing practical and economical solutions to many engineering situations. Trusses consist of straight members connected at joints (for example, see Figure 1). Note that truss members are connected at their extremities only: thus no truss members are continuous through a joint.

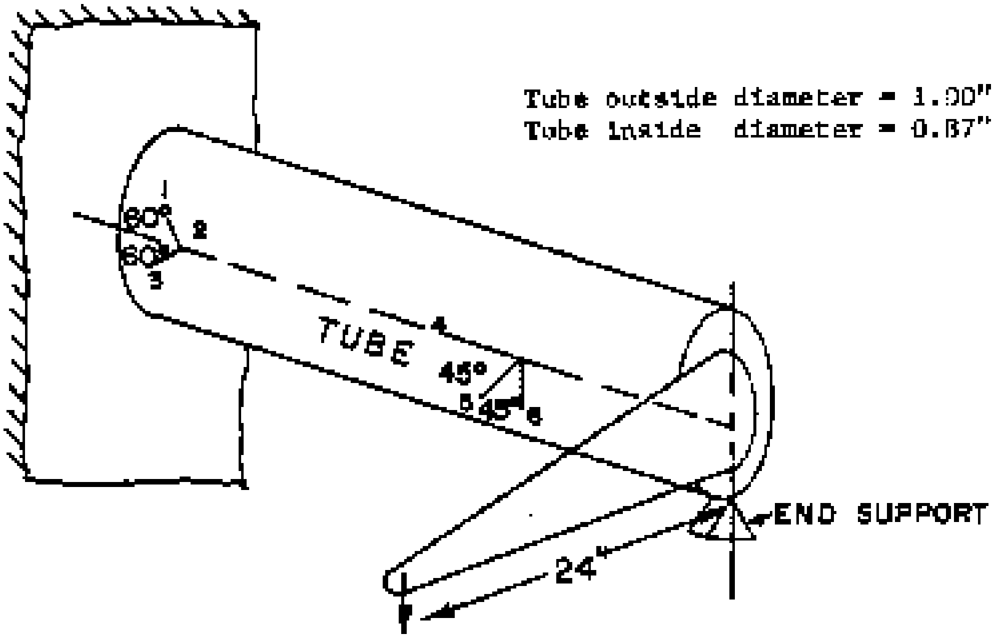


Figure 11.10 Relatively simple engineering component subjected to a complex loading condition

In general, truss members are slender and can support little lateral load. Therefore, major loads must be applied to the various joints and not the members themselves. Often the weights of truss members are assumed to be applied only at the joints (half the weight at each joint). In addition, even though the joints are actually rivets or welds, it is customary to assume that the truss members are pinned together (i.e., the force acting at the end of each truss member is a single force with no couple). Each truss member may then be treated as a two force member and the entire truss is treated as a group of pins and two-force members.

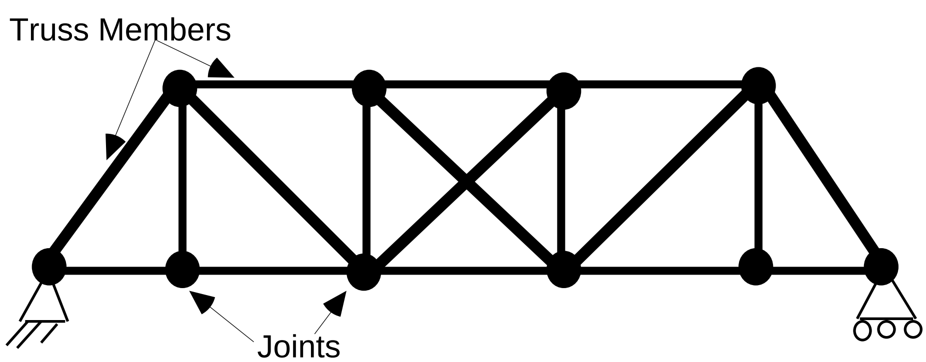


Figure 11.11 Example of a Simple Truss

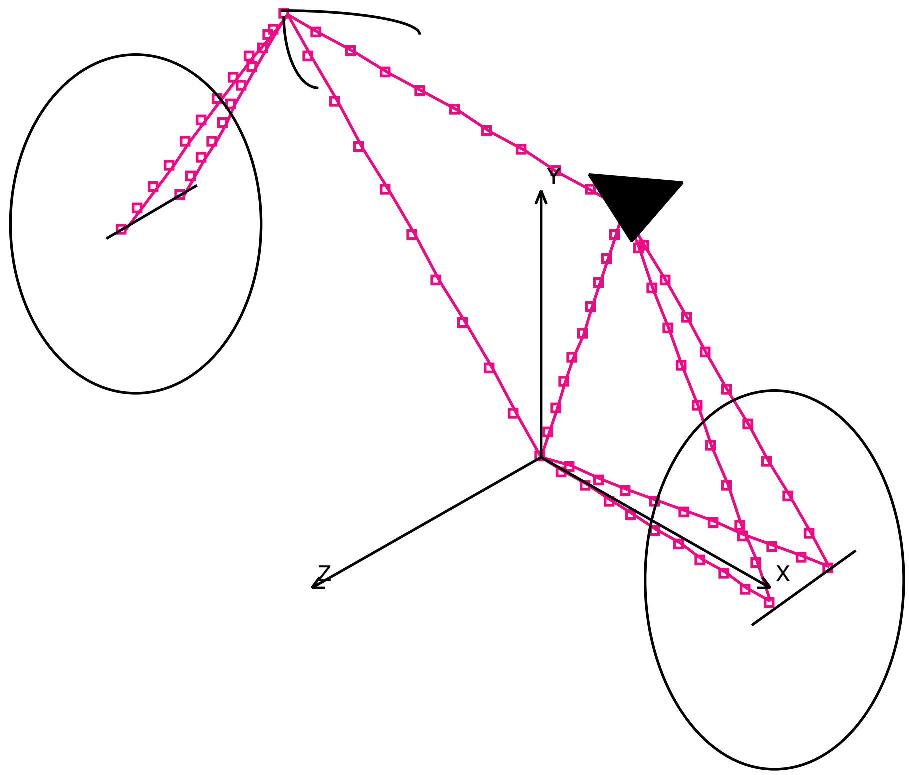


Figure 11.12 Illustration of a bicycle frame as a truss-like structure

A bicycle frame, on first inspection, appears to be an example of a truss (see Fig. 11.12) Each tube (truss member) is connected to the other at a joint, the principal loads are applied at joints (e.g., seat, steering head, and bottom bracket), and the reaction loads are carried at joints as well (e.g., front and rear axles). Although the joints are not pinned, a reasonable first approximation for analyzing forces, deflections, and stresses in the various tubes of the bicycle frame might be made using a simple truss analysis.

Forces in various truss members can be found using such analysis techniques as the method of joints or the method of sections. Deflections at any given joint may be found by using such analysis techniques as the unit load method of virtual work.

An example of the use of the method of joint to solve for the axial loads in each truss member is as follows. For the simple truss shown in Figure 11.13 the first step is to calculate the reactions at joints C and D. In this case, $\sum F = 0$ and $\sum M = 0$ such that

$$\sum M_C = 0 = PL - R_D L \qquad R_D = P \downarrow \tag{11.15}$$

and

$$\begin{aligned} \sum F = 0 \qquad & \sum F_x = 0 = -P + R_{xC} \qquad R_{xC} = P \rightarrow \\ & \sum F_y = 0 = -2P - P + R_{yC} \qquad R_{yC} = 3P \uparrow \end{aligned} \tag{11.16}$$

The resulting free body diagram is shown in Fig. 11.14

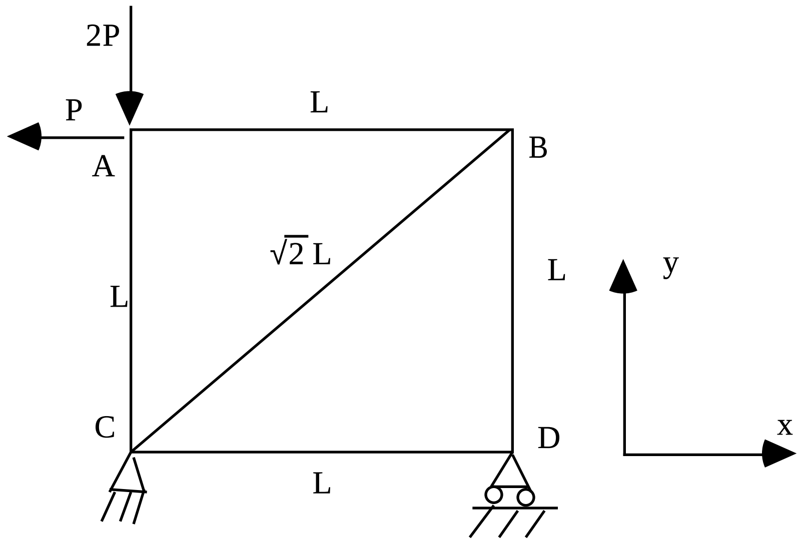
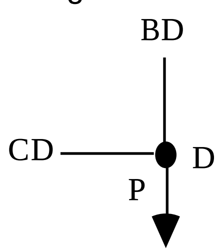


Figure 11.13 Example of a simple truss

Using the method of joints, $\sum F = 0$ at joint D such that



$$\sum F = 0 \qquad \begin{aligned} \sum F_x &= 0 = -F_{CD} \qquad F_{CD} = 0 \rightarrow \\ \sum F_y &= 0 = P + F_{BD} \qquad F_{BD} = P \uparrow \end{aligned} \tag{11.17}$$

and since F_{BD} pulls on the joint, then the joint must pull back on the member so member BD is in tension.

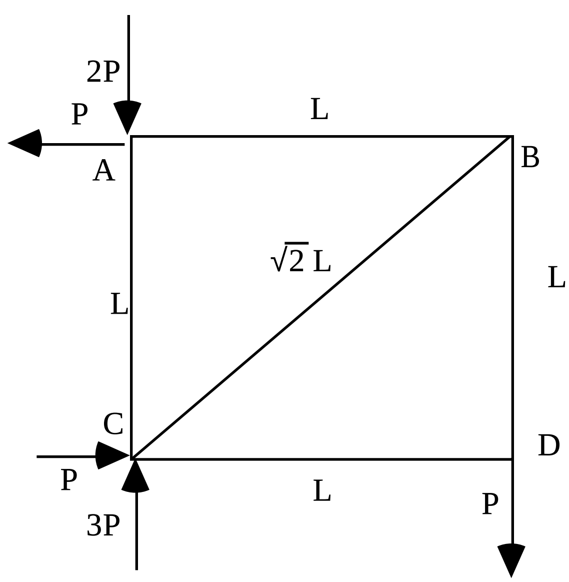
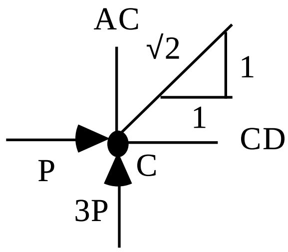


Figure 11.13 Free body diagram for simple truss

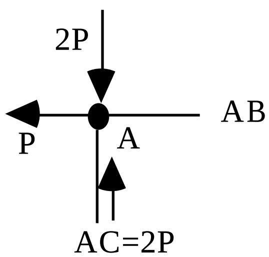
Using the method of joints, $\sum F$=0 at joint C such that



$$\sum F = 0 \qquad \sum F_x = 0 = P - \frac{1}{\sqrt{2}} F_{BC} \qquad F_{BC} = \sqrt{2} P \leftarrow$$
$$\sum F_y = 0 = 3P - \frac{1}{\sqrt{2}} F_{BC} - F_{AC} \qquad F_{AC} = P \downarrow \tag{11.18}$$

Since F_{AC} pushes on the joint, then the joint must push back on the member so member AC is in compression. Furthermore, since F_{CB} pushes on the joint, then the joint must push back on the member so member CB is in compression.

Finally, using the method of joints, $\sum F$=0 at joint A such that



$$\sum F = 0 \qquad \sum F_x = 0 = -P + F_{AB} \qquad F_{AB} = P \rightarrow$$
$$\sum F_y = 0 = -2P + 2P \qquad \textit{checks} \tag{11.19}$$

Since F_{AB} pulls on the joint, then the joint must pull back on the member so member AB is in tension.

The summary of the member forces is shown in Table 11.1.

Although finding deflections in complex structures is more involved than finding deflections in simple components, it is not difficult. A useful technique is the unit load method in which the displacements can be found from simple deflection equations at joints which do not have forces acting on them. The unit load method works for linearly elastic materials and superposition applies.

Table 11.1 Summary of Truss Member Forces

Member	Force
AB	P (tension)
AC	2P (compression)
BC	√2P (compression)
BD	P (tension)
CD	0

For axially loaded members, the displacement is:

$$\Delta_N = \int \frac{N_U N_L}{EA} dx \tag{11.20}$$

where N_U is the axial force in the member due to a unit load applied at the point and direction of interest, N_L is the actual force in the member due to the actual applied load on the structure, E and A are the elastic modulus and cross sectional area of the individual member. The integral sign signifies that the calculated quantities for each member are summed via integration to give the final total deflection at the point and direction of interest.

For members subjected to bending moments, the displacement is:

$$\Delta_M = \int \frac{M_U M_L}{EI} dx \tag{11.21}$$

where M_U is the bending moment in the member due to a unit load applied at the point and direction of interest, M_L is the actual bending moment in the member due to the actual applied load on the structure, E and I are the elastic modulus and cross sectional moment of inertia of the individual member. The integral sign signifies that the calculated quantities for each member are summed via integration to give the final total deflection at the point and direction of interest.

For members subjected to torsion, the displacement is:

$$\Delta_T = \int \frac{T_U T_L}{GJ} dx \tag{11.22}$$

where T_U is the torque in the member due to a unit load applied at the point and direction of interest, T_L is the actual torque in the member due to the actual applied load on the structure, G and J are the shear modulus and polar moment of inertia of the individual member. The integral sign signifies that the calculated quantities for each member are summed via integration to give the final total deflection at the point and direction of interest.

For members subjected to transverse shear, the displacement is:

$$\Delta_v = \int \frac{V_U V_L}{GA} dx \tag{11.23}$$

where V_U is the transverse shear in the member due to a unit load applied at the point and direction of interest, V_L is the actual transverse shear in the member due to the actual applied load on the structure, G and A are the shear modulus and cross sectional area of the individual member. The integral sign signifies that the calculated quantities for each member are summed via integration to give the final total deflection at the point and direction of interest.

The total deflection due to each of these contributions can then be found by adding the individual contribution such that

$$\Delta_t = \int \frac{N_U N_L}{EA} dx + \int \frac{M_U M_L}{EI} dx + \int \frac{V_U V_L}{GA} dx + \int \frac{T_U T_L}{GJ} dx \qquad (11.24)$$

An example of the unit load method applied to the simple truss example is shown in Fig. 11.14 in which only the axial loading contributions are required since truss members are pinned and no bending moments, transverse shear, or torque can be carried in the members.

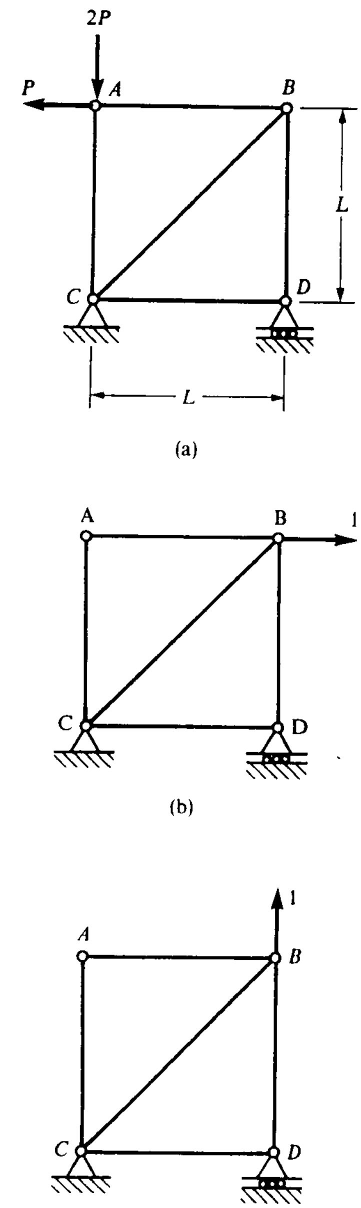


Fig. 10-4 Example 1. Displacements of a truss by the unit-load method

Example 1

The truss pictured in Fig. 10-4a is subjected to loads P and $2P$ at joint A. All members of the truss are assumed to be prismatic and to have the same axial rigidity EA. Calculate the horizontal and vertical displacements of joint B of the truss using the unit-load method.

Because the loads act only at the joints, the axial force in each member is constant throughout the length of the member. Therefore, we can use Eq. (10-5) to determine the desired deflections. It is helpful to record the calculations in a systematic manner, as shown in Table 10-2. The first two columns in the table identify the members of the truss and their lengths. The axial forces N_L, obtained by static-equilibrium analysis of the truss shown in Fig. 10-4a, are listed in column 3 of the table (tensile forces are positive).

Table 10-2 Calculations for Example 1

(1) Member	(2) Length	(3) N_L	(4) N_U	(5) $N_U N_L L$	(6) N_U	(7) $N_U N_L L$
AB	L	P	0	0	0	0
AC	L	$-2P$	0	0	0	0
BD	L	P	-1	$-PL$	1	PL
CD	L	0	0	0	0	0
CB	$\sqrt{2}L$	$-\sqrt{2}P$	$\sqrt{2}$	$-2.828PL$	0	0
				$-3.828PL$		PL

In order to find the horizontal displacement δ_h of joint B, we introduce a horizontal unit load on the structure at B (see Fig. 10-4b). The axial forces N_U produced by this unit load are given in column 4 of the table; again, tensile forces are positive. Next, the products $N_U N_L L$ are calculated for each member and summed (column 5). Dividing this result by EA gives the desired displacement (see Eq. 10-5):

$$\delta_h = -3.828\frac{PL}{EA}$$

The negative sign in this expression means that the displacement is in the direction opposite to the direction of the unit load; that is, toward the left.

The same general procedure is used to find the vertical displacement δ_v of joint B. The corresponding unit load (taken as positive when upward) is portrayed in Fig. 10-4c, and the axial forces N_U for this loading condition are listed in column 6 of the table. In the last column, the products $N_U N_L L$ are calculated and summed. Finally, dividing the sum by EA yields

$$\delta_v = \frac{PL}{EA}$$

Because this result is positive, we know that the vertical displacement of joint B produced by the loads P and $2P$ is upward.

Figure 11.14 Example of application of unit load method to find a deflection in a simple truss