12. Pressure Vessels: Combined Stresses

Cylindrical or spherical pressure vessels (e.g., hydraulic cylinders, gun barrels, pipes, boilers and tanks) are commonly used in industry to carry both liquid s and gases under pressure. When the pressure vessel is exposed to this pressure, the material comprising the vessel is subjected to pressure loading, and hence stresses, from all directions. The normal stresses resulting from this pressure are functions of the radius of the element under consideration, the shape of the pressure vessel (i.e., open ended cylinder, closed end cylinder, or sphere) as well as the applied pressure.

Two types of analysis are commonly applied to pressure vessels. The most common method is based on a simple mechanics approach and is applicable to "thin wall" pressure vessels which by definition have a ratio of inner radius, r, to wall thickness, t, of r/t≥10. The second method is based on elasticity solution and is always applicable regardless of the r/t ratio and can be referred to as the solution for "thick wall" pressure vessels. Both types of analysis are discussed here, although for most engineering applications, the thin wall pressure vessel can be used.

Thin-Walled Pressure Vessels

Several assumptions are made in this method.

1) Plane sections remain plane
2) r/t ≥ 10 with t being uniform and constant
3) The applied pressure, p, is the gage pressure (note that p is the difference between the absolute pressure and the atmospheric pressure)
4) Material is linear-elastic, isotropic and homogeneous.
5) Stress distributions throughout the wall thickness will not vary
6) Element of interest is remote from the end of the cylinder and other geometric discontinuities.
7) Working fluid has negligible weight

Cylindrical Vessels: A cylindrical pressure with wall thickness, t, and inner radius, r, is considered, (see Figure 12.1). A gauge pressure , p, exists within the vessel by the working fluid (gas or liquid). For an element sufficiently removed from the ends of the cylinder and oriented as shown in Figure 12.1, two types of normal stresses are generated: hoop, σ_h, and axial, σ_a, that both exhibit tension of the material.

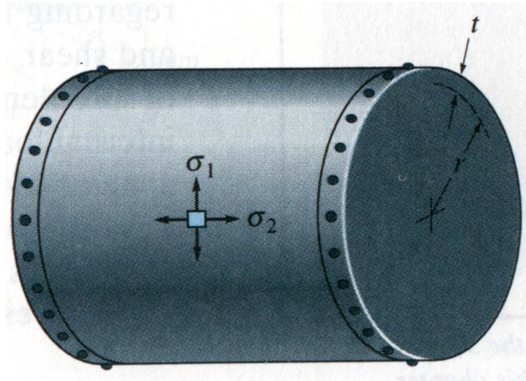


Figure 12.1 Cylindrical Thin-Walled Pressure Vessel

For the hoop stress, consider the pressure vessel section by planes sectioned by planes a, b, and c for Figure 12.2. A free body diagram of a half segment along with the pressurized working fluid is shown in Fig. 12.3 Note that only the loading in the x-direction is shown and that the internal reactions in the material are due to hoop stress acting on incremental areas, A, produced by the pressure acting on projected area, A_p. For equilibrium in the x-direction we sum forces on the incremental segment of width dy to be equal to zero such that:

$$\sum F_x = 0$$

$$2[\sigma_h A] - pA_p = 0 = 2[\sigma_h \text{ t dy}] - \text{p 2r dy}$$

or solving for σ_h (12.1)

$$\sigma_h = \frac{pr}{t}$$

where dy = incremental length, t = wall thickness, r = inner radius, p = gauge pressure, and σ_h is the hoop stress.

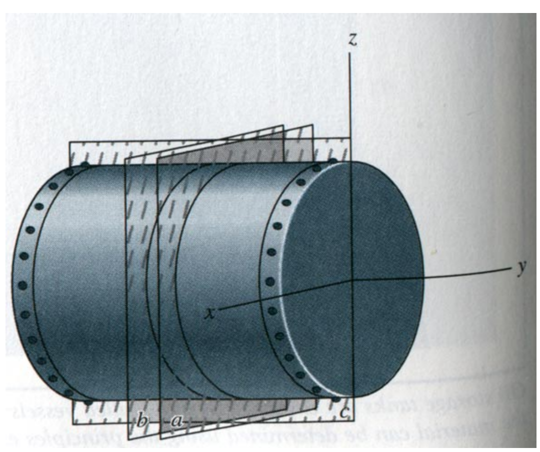


Figure 12.2 Cylindrical Thin-Walled Pressure Vessel Showing Coordinate Axes and Cutting Planes (a, b, and c)

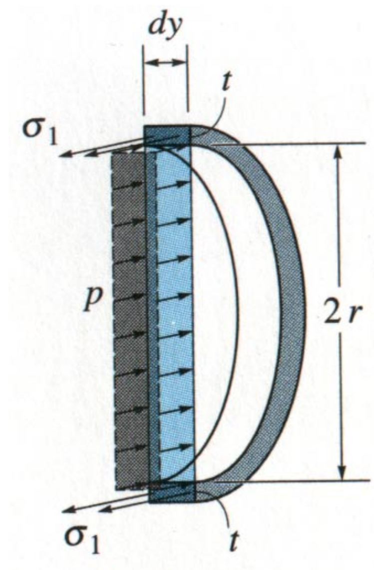


Figure 12.3 Free-Body Diagram of Segment of Cylindrical Thin-Walled Pressure Vessel Showing Pressure and Internal Hoop Stresses

For the axial stress, consider the left portion of section b of the cylindrical pressure vessel shown in Figure 12.2. A free body diagram of a half segment along with the pressurized working fluid is shown in Fig. 12.4 Note that the axial stress acts uniformly throughout the wall and the pressure acts on the endcap of the cylinder. For equilibrium in the y-direction we sum forces such that:

$$\sum F_y = 0$$

$$\sigma_a A - pA_e = 0 = \sigma_a \pi\left(r_o^2 - r^2\right) - \mathrm{p}\ \pi r^2$$

or solving for σ_a

$$\sigma_a = \frac{\mathrm{p}\ \pi r^2}{\pi\left(r_o^2 - r^2\right)}$$

substituting $\mathrm{r_o} = \mathrm{r} + \mathrm{t}$ gives (12.2)

$$\sigma_a = \frac{\mathrm{p}\ \pi r^2}{\pi\left(\left[\mathrm{r}+\mathrm{t}\right]^2 - r^2\right)} = \frac{\mathrm{p}\ \pi r^2}{\pi\left(r^2 + 2rt + t^2 - r^2\right)} = \frac{\mathrm{p}\ r^2}{\left(2rt + t^2\right)}$$

since this is a thin wall with a small t,t^2 is smaller and can be neglected such that after simplification

$$\sigma_a = \frac{\mathrm{p}\ r}{2t}$$

where $\mathrm{r_o}$ = inner radius and σ_a is the axial stress.

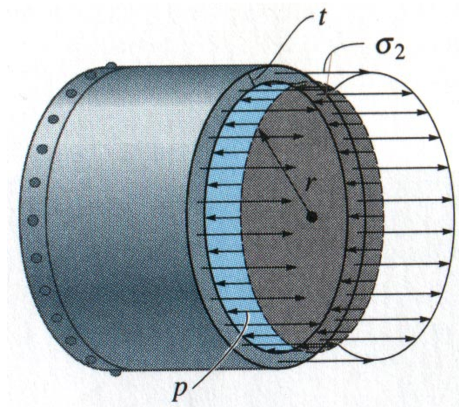


Figure 12.4 Free-Body Diagram of End Section of Cylindrical Thin-Walled Pressure Vessel Showing Pressure and Internal Axial Stresses

Note that in Equations 12.1 and 12.2, the hoop stress is twice as large as the axial stress. Consequently, when fabricating cylindrical pressure vessels from rolled-formed plates, the longitudinal joints must be designed to carry twice as much stress as the circumferential joints.

Spherical Vessels: A spherical pressure vessel can be analyzed in a similar manner as for the cylindrical pressure vessel. As shown in Figure 12-5, the "axial" stress results from the action of the pressure acting on the projected area of the sphere such that

$$\sum F_y = 0$$

$$\sigma_a A - pA_e = 0 = \sigma_a \pi\left(r_o^2 - r^2\right) - \text{p}\ \pi r^2$$

or solving for σ_a

$$\sigma_a = \frac{\text{p}\ \pi r^2}{\pi\left(r_o^2 - r^2\right)}$$

substituting $\text{r}_\text{o} = \text{r} + \text{t}$ gives (12.3)

$$\sigma_a = \frac{\text{p}\ \pi r^2}{\pi\left([\text{r}+\text{t}]^2 - r^2\right)} = \frac{\text{p}\ \pi r^2}{\pi\left(r^2 + 2rt + t^2 - r^2\right)} = \frac{\text{p}\ r^2}{\left(2rt + t^2\right)}$$

since this is a thin wall with a small t,t^2 is smaller and can be neglected such that after simplification

$$\sigma_a = \frac{\text{p}\ r}{2t} = \sigma_h$$

Note that for the spherical pressure vessel, the hoop and axial stresses are equal and are one half of the hoop stress in the cylindrical pressure vessel. This makes the spherical pressure vessel a more "efficient" pressure vessel geometry.

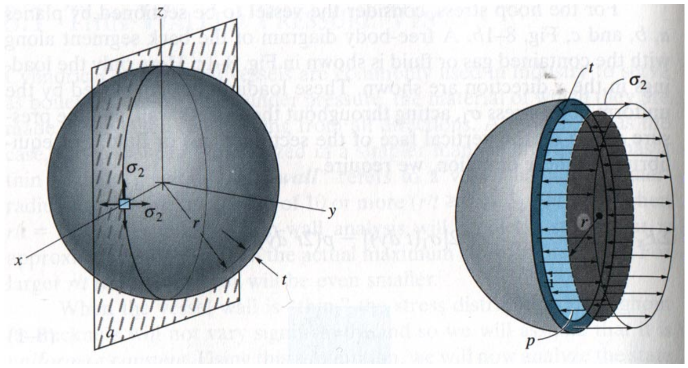


Figure 12.5 Free-Body Diagram of End Section of Spherical Thin-Walled Pressure Vessel Showing Pressure and Internal Hoop and Axial Stresses

The analyses of Equations 12.1 to 12.3 indicate that an element in either a cylindrical or a spherical pressure vessel is subjected to biaxial stress (i.e., a normal stress existing in only two directions). In reality, the element is subjected to a radial stress, $\sigma_r$which acts along a radial line. The stress has a compressive value equal to the pressure, p, at the inner wall, and decreases through the wall to zero at the outer wall (plane stress condition) since the gage pressure there is zero. For thin walled pressure vessels, the radial component is assumed to equal zero throughout the wall since the limiting assumption of r/t=10 results in σ_h being 10 times greater than σ_r=p and σ_a being 5 time greater than σ_r=p. Note also that the three normal stresses are principal stresses and can be used directly to determine failure criteria.

Note that the relations of Equation 12.1 to 12.3 are for internal gauge pressures only. If the pressure vessel is subjected to an external pressure, it may cause the pressure vessel to become unstable and collapse may occur by buckling of the wall.

Thick-Walled Pressure Vessels

Closed-form, analytical solutions of stress states can be derived using methods developed in a special branch of engineering mechanics called elasticity. Elasticity methods are beyond the scope of the course although elasticity solutions are mathematically exact for the specified boundary conditions are particular problems. For cylindrical pressure vessels subjected to an internal gage pressure only the following relations result:

$$\sigma_h = \frac{r_i^2 p}{\left(r_o^2 - r_i^2\right)}\left(1 + \frac{r_o^2}{r^2}\right)$$

$$\sigma_a = \frac{r_i^2 p}{\left(r_o^2 - r_i^2\right)} \qquad (12.4)$$

$$\sigma_r = \frac{r_i^2 p}{\left(r_o^2 - r_i^2\right)}\left(1 - \frac{r_o^2}{r^2}\right)$$

where r_o=outer radius, r_i=inner radius, and r is the radial variable. Equations 12.4 apply for any wall thickness and are not restricted to a particular r/t ratio as are the Equations 12.1 and 12.2. Note that the hoop and radial stresses(σ_h and σ_r) are functions of r (i.e. vary through the wall thickness) and that the axial stress, σ_a, is independent of r (i.e., is constant through the wall thickness. Figure 12.6 shows the stress distributions through the wall thickness for the hoop and radial stresses. Note that for the radial stress distributions, the maximum and minimum values occur, respectively, at the outer wall (σ_r=0) and at the (σ_r=-p) as noted already for the thin walled pressure vessel.

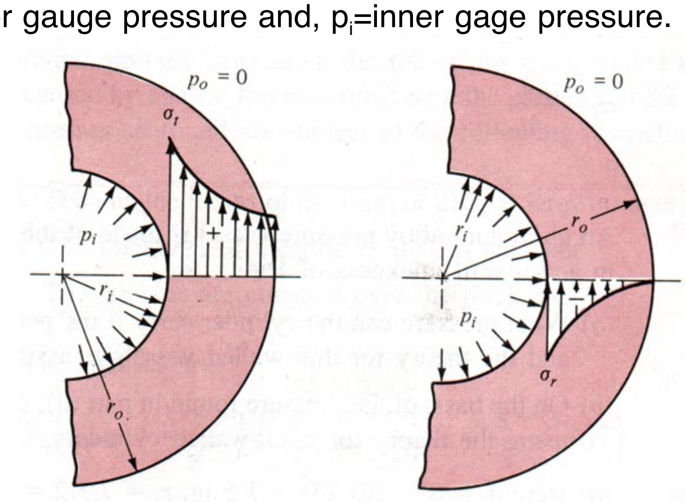
Equations 12.4 can be generalized for the case of internal and external pressures such that

$$\sigma_h = \frac{r_i^2 p_i - r_o^2 p_o - r_i^2 r_o^2 \left(p_o - p_i\right)/r^2}{\left(r_o^2 - r_i^2\right)}$$

$$\sigma_h = \frac{r_i^2 p_i - r_o^2 p_o}{\left(r_o^2 - r_i^2\right)} \qquad (12.5)$$

$$\sigma_h = \frac{r_i^2 p_i - r_o^2 p_o + r_i^2 r_o^2 \left(p_o - p_i\right)/r^2}{\left(r_o^2 - r_i^2\right)}$$

where p_o=is the outer gauge pressure and, p_i=inner gage pressure.



a) hoop stress b) radial stress

Figure 12.6 Stress distributions of hoop and radial stresses

Combined Loading

Typical formulae for stresses in mechanics of materials are developed for specific conditions. For example

$$\text{Axial Loading, } \sigma = \frac{P_N}{A_N}$$
$$\text{Beam Bending, } \sigma = \frac{My}{I} \text{ and } \tau = \frac{VQ}{It}$$
$$\text{Direct Shear, } \tau = \frac{P_T}{A_T} \qquad (12.6)$$
$$\text{Torsional Shear, } \tau = \frac{Tr}{J}$$
$$\text{Pressure Vessels, Shear, } \sigma_h = \frac{p\,r}{t},\ \sigma_a = \frac{p\,r}{2t},\ \sigma_r = 0$$

Often, the cross section of a member is subjected to several types of loadings simultaneously and as a result the method of superposition can be applied to determine the resultant stress distribution caused by the loads. In superposition, the stress distribution due to each loading is first determined, and then these distributions are superimposed to determine the resultant stress distributions. Note that only stresses of the same type and in the same direction can be superimposed. The principle of superposition can be used for the purpose provided that a linear relationship exists between the stress and the loads. In addition, the geometry of the member should not undergo significant change when the loads are applied. This is necessary in order to endure that the stress produced by one load is not related to the stress produced by any other loads. The following procedure is taken from (Hibbeler, 1997)

PROCEDURE FOR ANALYSIS

The following procedure provides a general means for establishing the normal and shear stress components at a point in a member when the member is subjected to several different types of loadings simultaneously. It will be assumed the material is homogeneous and behaves in a linear-elastic manner. Also, Saint-Venant's principle requires that the point where the stress is to be determined is far removed from any discontinuities in the cross section or points of applied load.

Internal Loadings. Section the member perpendicular to its axis at the point where the stress is to be determined. Use the necessary free-body diagrams and equations of equilibrium to obtain the resultant internal normal and shear force components and the bending and torsional moment components. The force components should act through the *centroid* of the cross section, and the moment components should be computed about *centroidal axes*, which represent the principal axes of inertia for the cross section.

Stress Components. Compute the stress component associated with *each* internal loading. For each case, represent the effect either as a distribution of stress acting over the entire cross-sectional area, or show the stress on an element of the material located at a specified point on the cross section.

NORMAL FORCE. The internal normal force is developed by a uniform normal-stress distribution determined from $\sigma = P/A$.

SHEAR FORCE. The internal shear force in a member that is subjected to bending is developed by a shear-stress distribution determined from the shear formula, $\tau = VQ/It$. Special care, however, must be exercised when applying this equation, as noted in Sec. 7.3.

BENDING MOMENT. For *straight members* the internal bending moment is developed by a normal-stress distribution that varies linearly from zero at the neutral axis to a maximum at the outer boundary of the member. The stress distribution is determined from the flexure formula, $\sigma = -My/I$. If the member is *curved*, the stress distribution is nonlinear and is determined from $\sigma = My/[Ae(R - y)]$.

TORSIONAL MOMENT. For circular shafts and tubes the internal torsional moment is developed by a shear-stress distribution that varies linearly from the central axis of the shaft to a maximum at the shaft's outer boundary. The shear-stress distribution is determined from the torsional formula, $\tau = T\rho/J$. If the member is a closed thin-walled tube, use $\tau = T/2A_m t$.

THIN-WALLED PRESSURE VESSELS. If the vessel is a thin-walled cylinder, the internal pressure p will cause a biaxial state of stress in the material such that the hoop or circumferential stress component is $\sigma_1 = pr/t$ and the longitudinal stress component is $\sigma_2 = pr/2t$. If the vessel is a thin-walled sphere, then the biaxial state of stress is represented by two equivalent components, each having a magnitude of $\sigma_2 = pr/2t$.

Superposition. Once the normal and shear stress components for each loading case have been calculated, use the principle of superposition and determine the resultant normal and shear stress components. Represent the results on an element of material located at the point, or show the results as a distribution of stress acting over the member's cross-sectional area.