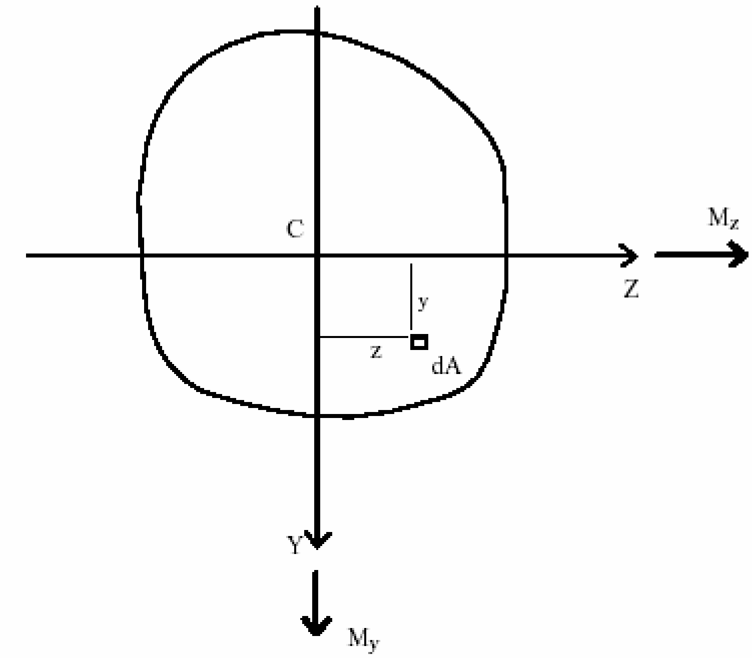
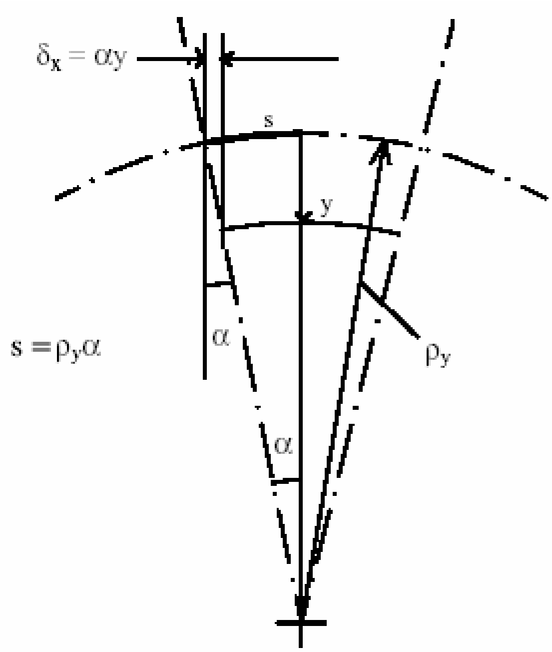
Bending of Beams with Unsymmetrical Sections



C = centroid of section

Assume that CZ is a neutral axis.

Hence, if $M_z > 0$, dA has negative stress. From the diagram below, we have:



$$\delta_x = \alpha y \quad \text{and} \quad s = \alpha \rho_y$$

$$\varepsilon_x = \frac{\delta_x}{s} = -\frac{y}{\rho_y}$$

and

$$\sigma_x = -\frac{Ey}{\rho_y} = -\kappa_y Ey$$

if M_z is the only load, we have:

$$\int \sigma_x dA = 0$$

$$-\kappa_y E \int ydA = 0$$

or

$$\int ydA = 0$$

hence the neutral axis passes through the centroid C.
A similar result holds for M_y and the Y axis.

Moment equilibrium about the Z axis:

$$-\int \sigma_x ydA = M_z$$

$$\kappa_y E \int y^2 dA = M_z$$

$$M_z = \kappa_y EI_z$$

and about the Y axis we have:

$$-\int \sigma_x zdA = M_y$$

$$\kappa_y E \int yz \ dA = M_y$$

$$M_y = \kappa_y EI_{yz}$$

and, if CZ is a **neutral** axis, we will have moments such that:

$$\frac{M_y}{M_z} = -\frac{I_{yz}}{I_z}$$

Similarly, if CY is a **neutra**l axis, we will have moments such that:

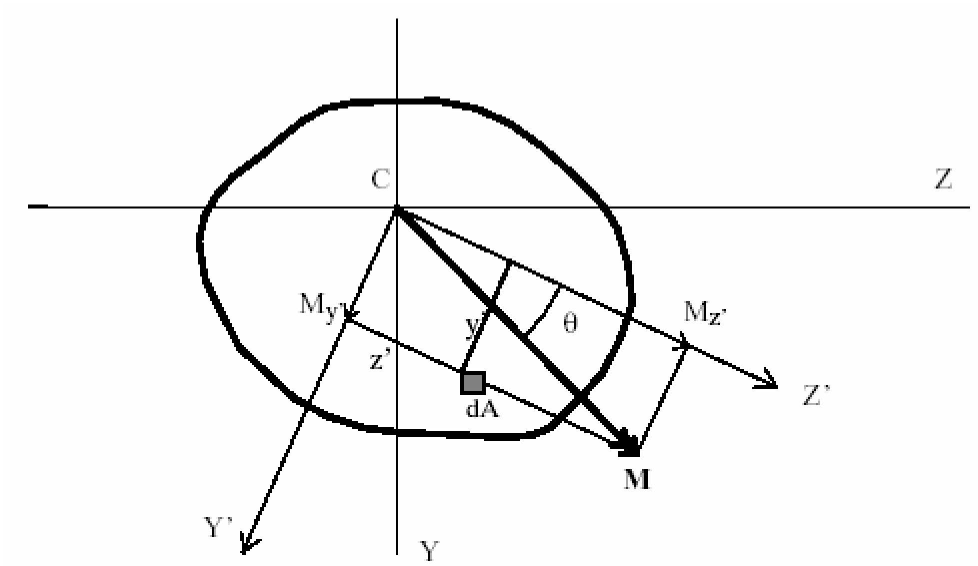
$$\frac{M_z}{M_y} = -\frac{I_{yz}}{I_z}$$

However, if CZ is a **principal** axis, $I_{yz} = 0$.
Therefore, if CZ is also the neutral axis, we have $M_y = 0$, i.e. bending takes place in the XY–plane just as for symmetrical bending.

Therefore the plane of the bending moment is perpendicular to the neutral surface only if the Y and Z axes are principal axes.

Hence, we can tackle bending of beams of non–symmetric cross section by:

(1) finding the principal axes of the section
(2) resolving moment M into components in the principal axis directions
(3) calculating stresses and deflections in each direction
(4) superimpose stresses and deflections to get the final result



Let *Y'* and *Z'* be the principal axes and let **M** be the bending moment vector. Resolving into components with respect to the principal axes we get:

$$M_{y'} = M \sin\theta$$
$$M_{z'} = M \cos\theta$$

If $I_{y'}$, $I_{z'}$ are the principal moments of inertia

$$\sigma_x = \frac{M_{y'} z'}{I_{y'}} - \frac{M_{z'} y'}{I_{z'}}$$
$$\sigma_x = \frac{M \sin(\theta) z'}{I_{y'}} - \frac{M \cos(\theta) y'}{I_{z'}}$$

For the neutral axis, $\sigma_x = 0$ by definition, hence as the point (y',z') lies on the neutral axis in this case, we have the neutral axis at angle β with **respect to the principal axis CZ**

and hence $$\tan\beta = \frac{I_{z'}}{I_{y'}}\tan\theta$$

Note that $\beta \neq 0$ in general.

The above method is most useful when the principal axes are known or can be found easily by calculation or inspection. The method is also useful for finding deflections (see below). It is also possible to calculate stresses with respect to a set of non–principal axes.

$$\sigma_x = \frac{\left(M_y I_z + M_z I_{yz}\right)z - \left(M_z I_y + M_y I_{yz}\right)y}{I_y I_z - I_{yz}^2}$$

The neutral axis is at an angle ϕ given by:

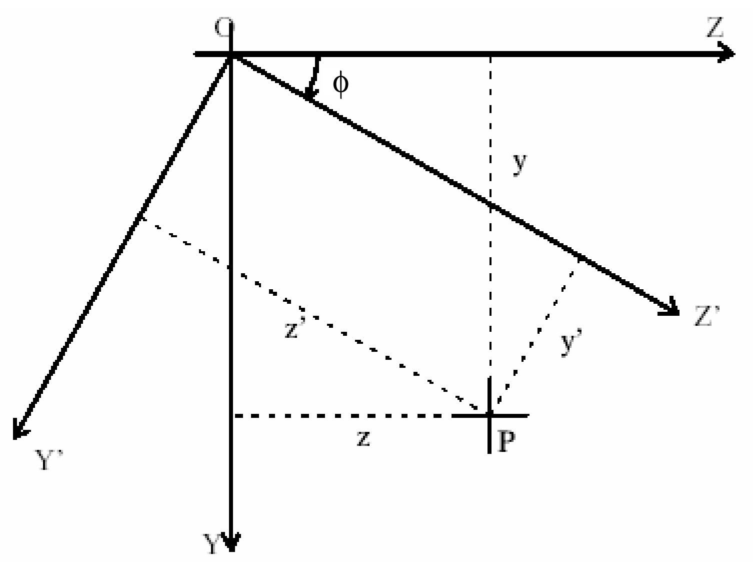
$$\tan\phi = \frac{M_y I_z + M_z I_{yz}}{M_z I_y + M_y I_{yz}}$$

This method is useful if the principal axes are not easily found but the components I_y, I_z and I_{yz} of the inertia tensor can be readily determined.

Deflections:

Using the first method described above, deflections can be found easily by resolving the applied lateral forces into components parallel to the principal axes and separately calculating the deflection components parallel to these axes. The total deflection at any point along the beam is then found by combining the components at that point into a resultant deflection vector. Note that the resulting deflection will be perpendicular to the neutral axis of the section at that point.

Rotation Transformations:



If *y*, *z* are the coordinates of point P in the the system YZ shown above, then the coordinates of P in the system Y'Z' are:

$$y' = y\cos\phi - z\sin\phi$$
$$z' = y\sin\phi + z\cos\phi$$

This transformation is useful in finding the coordinates of points with respect to the principal axes of a section.

Problems on Unsymmetrical Beams

1. An angle section with equal legs is subject to a bending moment vector M having its direction along the 1–1 direction as shown in Table E–4 in your textbook. Calculate the maximum tensile stress σ_t and the maximum compressive stress σ_c if the angle is a L 6x6x3/4 steel section and |M| = 20000 in.lb.

(σ_t = 3450 psi : σ_c = –3080 psi).

2. An angle section with unequal legs is subjected to a bending Moment **M** having its direction along the 1–1 direction as shown in Table E–5 in your textbook. Calculate the maximum tensile stress σ_t and the maximum compressive stress σ_c if the angle is a L 8x6x1 and |M| = 25000 lb.in.

(σ_t = 1840 psi : σ_c = –1860 psi).

3. Solve the previous problem for a L 7x4x1/2 section and |M| = 15000 lb. in.

(σ_t = 2950 psi : σ_c = –2930 psi).

See next page for section properties needed in these problems.

Section properties for structural steel angle sections.

Designation	Weight per ft.	Area	Axis ZZ			Axis YY			Axis Y'Y'	
			I_{ZZ}	r_{ZZ}	d	I_{YY}	r_{YY}	c	r_{min}	tan α
in.	lb.	in^2	in^4	in.	in.	in^4	in.	in.	in.	
L6x6x3/4	28.7	8.44	28.2	1.83	1.78	28.2	1.83	1.78	1.17	1
L8x6x1	44.2	13	80.8	2.49	2.65	38.8	1.73	1.65	1.28	0.543
L7x4x1/2	17.9	5.25	26.7	2.25	2.42	6.53	1.11	0.917	0.872	0.335

1. Axes ZZ and YY are centroidal axes parallel to the legs of the section.
2. Distances c and d are measured from the centroid to the outside surfaces of the legs.
3. Axes Y'Y' and Z'Z' are the principal centroidal axes.
4. The moment of inertia for axis Y'Y' is given by $I_{Y'Y'} = Ar^2_{min}$.
5. The moment of inertia for axis Z'Z' is given by $I_{Z'Z'} = I_{YY} + I_{ZZ} - I_{Y'Y'}$.