ME 374, System Dynamics Analysis and Design
Homework 2

Distributed: 4/2/2012, Due: 4/13/2012

(There are 6 problems in this set.)

1. Consider the state equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

   (a) Determine eigenvalues, eigenvectors, and modal matrix of the state matrix. Can you find enough eigenvectors?

   (b) Find the state transition matrix $\Phi(t)$ through eigenvector expansion. Be careful about the formula you use.

   (c) Find the state transition matrix $\Phi(t)$ through

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots \tag{2}$$

   It should be identical to one you obtain from the eigenvector expansion. You may need to use the Taylor expansion for e^t given as

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \tag{3}$$

   (d) What is the solution of the state equation (1) with initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{4}$$

2. Find the state transition matrix by the method of eigenvector expansion, i.e.,

$$\Phi(t) = \mathbf{M}\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}\mathbf{M}^{-1} \tag{5}$$

   for the following two matrices

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \tag{6}$$

   Can we use equation (5) to find $\Phi(t)$ associated with $\mathbf{A}_2$? Why?

3. This is an exam problem that I gave in Winter Quarter of 2001. Consider the state equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7}$$

   where x_1 and x_2 are state variables. The state equation (7) is subjected to the following initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8}$$

(a) Determine the eigenvalues and eigenvectors of the state matrix.

(b) Find the modal matrix.

(c) Let $\Phi(t)$ be the state transition matrix. Determine $\Phi(t)$ through eigenvector expansion. Hint: A useful formula is the inversion of 2×2 matrix as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{9}$$

(d) Determine $\Phi(t)$ by explicitly working out the series

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \dots \tag{10}$$

In other words, you need to calculate $\mathbf{A}^2$, $\mathbf{A}^3$, and so on.

(e) Determine the solution of (7) subjected to the initial conditions in (8). Use the state transition matrix.

4. This is an exam problem of Spring Quarter of 2004. Consider the state equation

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11}$$

where x_1 and x_2 are state variables.

(a) Determine the stability of the system. Will the response of the system oscillate? Explain why.

(b) Let $\Phi(t)$ be the state transition matrix. Determine $\Phi(t)$ through eigenvector expansion. Hint: A useful formula is the inversion of 2×2 matrix as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{12}$$

(c) The state equation (11) is subjected to the following initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13}$$

Determine the solution of (11) through use of the state transition matrix.

5. This is an old exam problem from 2008 Spring Quarter. Consider the state equation

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{14}$$

where x_1, x_2 and x_3 are state variables. Please answer the following questions.

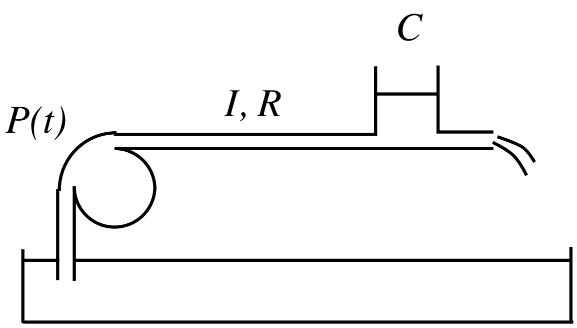

Figure 1: A fluid system

(a) The state matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

has three-fold eigenvalues with $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Find all independent eigenvectors corresponding to this eigenvalue.

(b) Find the modal matrix $\mathbf{M}$ associated with the state matrix $\mathbf{A}$. Does $\mathbf{M}^{-1}\mathbf{AM}$ lead to a Jordan form or not? Hint: The modal matrix $\mathbf{M}$ turns out to be a diagonal matrix. For a diagonal matrix, its inverse is given by

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix} \tag{16}$$

(c) Find the state transition matrix $\Phi(t)$.

(d) Determine the stability of the system. Please justify your answer.

6. Consider the fluid system shown in Fig. 1. The state variables are the pressure p_C of the tank and the flow rate Q_I in the long pipe. In addition, the input is the pump pressure $P(t)$. Moreover, the state equation of the system is given by

$$\frac{d}{dt}\begin{pmatrix} p_C \\ Q_I \end{pmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & -4 \end{bmatrix}\begin{pmatrix} p_C \\ Q_I \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} P(t) \tag{17}$$

Answer the following questions.

(a) Is this system stable?

(b) If the pump pressure $P(t)$ surges from 0 to a constant $\bar{P}$, the fluid level in tank C will finally reach to a new equilibrium position. Do you think the fluid level will change to the new equilibrium position exponentially or oscillatorily? Explain why.