$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) Eigenvalues:

$$\left| \underline{A} - \lambda \underline{I} \right| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 = 0 \quad , \quad \lambda_1 = \lambda_2 = 1$$

Eigenvector:

$$\left[\underline{A} - \lambda \underline{I} \right] \underset{\sim}{u}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underset{\sim}{u}_1 = \underset{\sim}{0} . \qquad \underset{\sim}{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Not Enough Eigenvectors

Modal Matrix

$$(\underline{A} - \lambda \underline{I}) \underset{\sim}{u}_2 = \underset{\sim}{u}_1 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} v_2 = 1 \\ 0 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} v_1 = \text{arbitrary} \\ v_2 = 1 \end{cases}$$

Choose $v_1 = 0$

$$\underset{\sim}{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence

$$\underline{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{M}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) State transition matrix

$$\underline{\Phi}(t) = \underline{M}\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix}\underline{M}^{-1} = \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}$$

(c) $$e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{1}{2!}\underline{A}^2t^2 + \frac{1}{3!}\underline{A}^3t^3 + \cdots$$

$$\underline{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \underline{A}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\underline{A}^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\underline{A}^4 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \cdots$$

$$\underline{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$e^{\underline{A}t} = \begin{bmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & t + 2\cdot\frac{t^2}{2!} + 3\cdot\frac{t^3}{3!} + \cdots \\ 0 & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}$$

(d) $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{t}(-1+3t) \\ e^{t} \end{bmatrix}$$