$$\underset{\sim}{\Phi}(t) = \underline{M} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \quad e^{\lambda_n t} \end{bmatrix} \underline{M}^{-1}$$

$$\underline{A}_1 = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}, \qquad \underline{A}_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

(a) For $\underline{A}_1$

Eigenvalues : $|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = \lambda(\lambda+7)+6 = \lambda^2 + 7\lambda + 6 = 0$

$\lambda_1 = -1, \quad \lambda_2 = -6$

Eigenvectors

$$(\underline{A} - \lambda_1 \underline{I})\underset{\sim}{u}_1 = \begin{bmatrix} +1 & 1 \\ -6 & -6 \end{bmatrix} \underset{\sim}{u}_1 = \underset{\sim}{0}, \qquad \underset{\sim}{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(\underline{A} - \lambda_2 \underline{I})\underset{\sim}{u}_2 = \begin{bmatrix} +6 & 1 \\ -6 & -1 \end{bmatrix} \underset{\sim}{u}_2 = \underset{\sim}{0}, \qquad \underset{\sim}{u}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

Modal Matrix

$$\underline{M} = \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix}, \qquad |\underline{M}| = -6+1 = -5$$

$$\underline{M}^{-1} = -\frac{1}{5}\begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6 & 1 \\ -1 & -1 \end{bmatrix}$$

State Transition Matrix

$$\Phi(t) = \begin{bmatrix} 1 & 1 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \left(\frac{1}{5}\right) \begin{bmatrix} 6 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix} \begin{bmatrix} 6e^{-t} & e^{-t} \\ e^{-6t} & -e^{-6t} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -6e^{-t} + e^{-6t} & -e^{-t} + e^{-6t} \\ 6e^{-t} - 6e^{-6t} & e^{-t} - 6e^{-6t} \end{bmatrix}$$

(b) for $\underline{A}_2$

Eigenvalues. $|\underline{A}_2 - \lambda \underline{I}| = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda(\lambda-2)+1 = (\lambda-1)^2 = 0$

$$\lambda_1 = \lambda_2 = 1$$

Eigenvectors:

$$(\underline{A}_2 - \lambda_1 \underline{I})\,\underset{\sim}{u}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \underset{\sim}{u}_1 = 0, \qquad \underset{\sim}{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(\underline{A}_2 - \lambda_2 \underline{I})\,\underset{\sim}{u}_2 = \underset{\sim}{u}_1$$

or

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Choose $x_1 = 0$, $x_2 = 1$ $\Rightarrow$ $\underset{\sim}{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Modal matrix

$$\underline{M} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad |\underline{M}| = 1, \quad \underline{M}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

State Transition matrix

$$\underline{\Phi}(t) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{t}(1+t) & te^{t} \\ e^{t} & e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}(1+t) & te^{t} \\ -te^{t} & (1-t)e^{t} \end{bmatrix}$$

Since $\underline{A}_2$ has repeated eigenvalues and $\underline{A}_2$ is NOT symmetric, $\underline{A}_2$ does not have enough eigenvectors to form the modal matrix. Hence

$$\underline{\Phi}(t) = \underline{M} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \underline{M}^{-1}$$