$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \underset{\sim}{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) Eigenvalues

$$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1) + 1 = \lambda^2 = 0$$

$$\lambda_1 = \lambda_2 = 0$$

Eigenvectors

$$(\underline{A} - \lambda_1 \underline{I})\underset{\sim}{u}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\underset{\sim}{u}_1 = 0. \qquad \underset{\sim}{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

only 1 Eigenvector

(b) Modal matrix

$$(\underline{A} - \lambda_2 \underline{I})\underset{\sim}{u}_2 = \underset{\sim}{u}_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

choose $x_2 = 0$, $x_1 = 1$ $\Rightarrow$ $\underset{\sim}{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\underline{M} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

(c) $\underline{\Phi}(t)$ by Eigenvector Expansion

det $\underline{M} = 1$, $\qquad \underline{M}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

$$\underline{\Phi}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & t e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} ; \quad e^{\lambda_1 t} = e^{\lambda_2 t} = 1$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} t & t-1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} t+1 & t \\ -t & 1-t \end{bmatrix}$$

(d) Series Expansion

$$\underline{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \underline{A}^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{A}^3 = \underline{A}^4 = \cdots = \underline{0}$$

Hence

$$e^{\underline{A}t} = \underline{I} + \underline{A}t + \underbrace{\frac{\underline{A}^2 t^2}{2!}}_{\to 0} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} t = \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

(e)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underline{\Phi}(t)\, \underset{\sim}{x}(0) = \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$