

Spring 2008, Prob. #1

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$(a) \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0, \quad \lambda = 1, 1, 1$$

$\lambda = 1,$

$$[A - \lambda I] \underline{u} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{aligned} c_2 &= 0 \\ c_1 &= \text{arbitrary} \\ c_3 &= \text{arbitrary} \end{aligned}$$

Hence

$$\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Need one more vector to form modal matrix

$$(A - \lambda I) \underline{u}_3 = \underline{u}_1$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{u}_3 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\underline{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{M}^{-1} A \underline{M} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Jordan form}$$

$$(c) \quad \underline{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & t e^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
 \Phi(t) &= e^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= e^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2t & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^t \begin{bmatrix} 1 & 2t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

(d) The system is unstable. At least one eigenvalue has Real part greater than zero.