Spring 2008, Prob. #2

$$\tau \frac{dx}{dt} + x = A f(t) \quad ; \quad \tau, A = \text{unknown}$$

(a) Transfer function

$$x(t) = X(s) e^{st}, \quad f(t) = F(s) e^{st}$$

$$H(s) \equiv \frac{X(s)}{F(s)} = \frac{A}{\tau s + 1}$$

(b) Input : $f(t) = \cos 3t + \sin 3t = Re\left[(1+j) e^{-3jt} \right]$

$$(1+j) = \sqrt{2}\, e^{j\pi/4}$$

$$(1+j) e^{-3jt} = \sqrt{2}\, e^{j(-3t + \frac{\pi}{4})}$$

Magnitude $= \sqrt{2}$, phase $= -3t + \frac{\pi}{4}$

(c) Output : $|H(s)| = 2\sqrt{2}$. $\measuredangle H(s) = \frac{\pi}{4}$

output magnitude $= 2\sqrt{2} \cdot \sqrt{2} = 4$

output phase $= \frac{\pi}{4} - 3t + \frac{\pi}{4} = -3t + \frac{\pi}{2}$

Hence

$$x(t) = Re\left[4 e^{j(-3t + \frac{\pi}{2})} \right] = Re\left[4j (\cos 3t - j \sin 3t) \right]$$

$$= 4 \sin 3t$$

(d) Parameters A & τ

With $f(t) = Re\left[(1+j)e^{-3jt}\right] \Rightarrow s = -3j$

$$H(s) = \frac{A}{1-j(3\tau)} = \frac{A}{1-j3\tau} \times \frac{1+j3\tau}{1+j\cdot 3\tau} = \frac{A}{1+9\tau^2}(1+j\cdot 3\tau)$$

Hence

$$\measuredangle H(s) = \tan^{-1} 3\tau = \frac{\pi}{4} \Rightarrow 3\tau = \tan\frac{\pi}{4} = 1 \Rightarrow \tau = 1/3$$

$$\Rightarrow \quad H(s) = \frac{A}{1+1}(1+j) = \frac{A}{2}(1+j)$$

$$|H(s)| = \frac{A}{2}\sqrt{2} = 2\sqrt{2} \Rightarrow A = 4$$