Problem 1

$$\underline{A}_1 = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \quad \underline{A}_2 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \quad \underline{A}_3 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

(a) Eigenvalues & Eigenvectors

$$|\underline{A}_1 - \lambda \underline{I}| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = \lambda(7+\lambda) + 6 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+6) = 0 \Rightarrow \underline{\lambda_1 = -1, \quad \lambda_2 = -6}_{\text{ANS.}}$$

✓ when $\lambda = -1$ $\quad (\underline{A}_1 - \lambda \underline{I})\underset{\sim}{x} = \underset{\sim}{0}$

$$\begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ -6x_1 - 6x_2 = 0 \end{cases} \Big\rangle \text{ Linearly dependent}$$

$$\Rightarrow x_1 = -x_2, \quad \text{Let } x_1 = c_1 \text{ (arbitary)}$$

$$\Rightarrow \begin{cases} x_1 = c_1 \\ x_2 = -c_1 \end{cases} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \xrightarrow{c_1 = 1} \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}_{\text{ANS.}}$$

✓ when $\lambda = -6$ $\quad (\underline{A}_1 - \lambda \underline{I})\underset{\sim}{x} = \underset{\sim}{0}$

$$\begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left\{ \begin{array}{l} 6x_1 + x_2 = 0 \\ -6x_1 - x_2 = 0 \end{array} \right\rangle \text{Linear dependent}$$

$$\Rightarrow 6x_1 = -x_2 \,, \quad \text{Let } x_1 = C_2 \text{ (arbitary)}$$

$$\Rightarrow \left\{ \begin{array}{l} x_1 = C_2 \\ x_2 = -6C_2 \end{array} \right. \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_2 \begin{Bmatrix} 1 \\ -6 \end{Bmatrix} \xrightarrow{C_2 = 1} \underset{\sim}{u}_2 = \begin{Bmatrix} 1 \\ -6 \end{Bmatrix}$$

ANS.

$$|\underline{A}_2 - \lambda \underline{I}| = \begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 16 = 0$$

$$\Rightarrow \lambda^2 = -16 \Rightarrow \underline{\lambda_1 = 4i \,, \quad \lambda_2 = -4i}$$

ANS.

✓ <u>when $\lambda = 4i$</u> $\quad (\underline{A}_2 - \lambda \underline{I}) \underset{\sim}{x} = \underset{\sim}{0}$

$$\begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left\{ \begin{array}{l} -4i x_1 + 4x_2 = 0 \\ -4x_1 - 4i x_2 = 0 \end{array} \right. \Rightarrow i x_1 = x_2 \,, \quad \text{Let } x_1 = C_1$$

$$\Rightarrow \left\{ \begin{array}{l} x_1 = C_1 \\ x_2 = C_1 i \end{array} \right. \Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ i \end{Bmatrix} \xrightarrow{C_1 = 1} \underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ i \end{Bmatrix}$$

ANS.

✓ <u>when $\lambda = -4i$</u> $\quad (\underset{\sim}{A}_2 - \lambda \underset{\sim}{I})\underset{\sim}{X} = \underset{\sim}{0}$

$$\begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 4ix_1 + 4x_2 = 0 \\ -4x_1 + 4ix_2 = 0 \end{cases} \Rightarrow ix_1 = -x_2, \quad \text{Let } x_1 = C_2$$

$$\Rightarrow \begin{cases} x_1 = C_2 \\ x_2 = -C_2 i \end{cases} \Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_2 \begin{Bmatrix} 1 \\ -i \end{Bmatrix} \xrightarrow{C_2 = 1} \underline{\underset{\sim}{u}_2 = \begin{Bmatrix} 1 \\ -i \end{Bmatrix}}$$

$$|\underset{\sim}{A}_3 - \lambda \underset{\sim}{I}| = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = -(2-\lambda)\lambda + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0 \quad \Rightarrow \underline{\lambda_1 = \lambda_2 = 1} \text{ ANS.} \quad (\text{repeated})$$

✓ <u>when $\lambda = 1$</u> $\quad (\underset{\sim}{A}_3 - \lambda \underset{\sim}{I})\underset{\sim}{X} = \underset{\sim}{0}$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2, \quad \text{Let } x_1 = C_1$$

$$\Rightarrow \begin{cases} x_1 = C_1 \\ x_2 = -C_1 \end{cases} \Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \xrightarrow{C_1 = 1} \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}} \text{ ANS.}$$

* only has one eigenvector!

(b)

```
clear all
close all

A1 = [0 1; -6 -7];
A2 = [0 4; -4 0];
A3 = [2 1; -1 0];

% D are eigenvelues (diagonal elements)
% V are corresponding eigenvectors (normalized)

[V,D]=eig(A1);

% V =
%     0.7071   -0.1644
%    -0.7071    0.9864
%
% D =
%     -1      0
%      0     -6

[V,D]=eig(A2);

% V =
%    0.7071                 0.7071
%         0 + 0.7071i            0 - 0.7071i
%
% D =
%         0 + 4.0000i            0
%         0                      0 - 4.0000i

[V,D]=eig(A3);

% V =
%     0.7071   -0.7071
%    -0.7071    0.7071
%
% D =
%     1.0000         0
%          0    1.0000
```

(C) For A_1, Modal Matrix

$$M_1 = [u_1, u_2] = \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix}$$

$$M_1^{-1} = \frac{1}{\det M_1} \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{-6+1} \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6/5 & 1/5 \\ -1/5 & -1/5 \end{bmatrix}$$

Diagonalize A_1:

$$M_1^{-1} A_1 M_1 = \begin{bmatrix} -6/5 & 1/5 \\ -1/5 & -1/5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$$

ANS.

For A_2, Modal Matrix

$$M_2 = [u_1, u_2] = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$M_2^{-1} = \frac{1}{\det M_2} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{-i-i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{bmatrix}$$

Diagonalize A_2:

$$M_2^{-1} A_2 M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix}$$

ANS.

For A_3, Modal Matrix

$$(\underline{A}_3 - \lambda_2 \underline{I})\, \underset{\sim}{u}_2 = u_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 1 \\ -x_1 - x_2 = -1 \end{cases} \Rightarrow x_1 = 1 - x_2, \quad \text{Let } x_1 = C_2$$

$$\Rightarrow \begin{cases} x_1 = C_2 \\ x_2 = 1 - C_2 \end{cases} \Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_2 \\ 1 - C_2 \end{Bmatrix} \xrightarrow{C_2 = 1} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

ANS.

Note: U_2 is NOT an eigenvector!

$$M_3 = [u_1, u_2] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M_3^{-1} = \frac{1}{\det M_3} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{0+1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Diagonalize A_3:

$$M_3^{-1} A_3 M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

ANS.

This is a Jordan Form

Problem 2:

(a) Ellipse: $F = 34x^2 - 24xy + 41y^2 = C$

This quadratic form can be:

$$F = \begin{pmatrix} x \\ y \end{pmatrix}^T A \begin{pmatrix} x \\ y \end{pmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Since A is symmetric, so $a_{12} = a_{22}$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Thus,

$$F = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{bmatrix} a_{11}x + a_{12}y \\ a_{12}x + a_{22}y \end{bmatrix}$$

$$= a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

$$= 34x^2 - 24xy + 41y^2$$

$$\Rightarrow a_{11} = 34; \quad a_{12} = -12; \quad a_{22} = 41$$

$$\Rightarrow A = \begin{bmatrix} 34 & -12 \\ -12 & 41 \end{bmatrix}$$

ANS.

(b) $(\underline{A} - \lambda \underline{I})\, \underset{\sim}{u} = \underset{\sim}{0}$

$$\det|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 34-\lambda & -12 \\ -12 & 41-\lambda \end{vmatrix} = \lambda^2 - 75\lambda + 1250 = 0$$

$$\Rightarrow (\lambda - 50)(\lambda - 25) = 0$$

$$\Rightarrow \underline{\lambda_1 = 50 \; ; \quad \lambda_2 = 25} \quad \text{ANS.}$$

✓ when $\lambda_1 = 50$ $\quad (\underline{A} - \lambda_1 \underline{I})\, \underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} -16 & -12 \\ -12 & -9 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} -16x_1 - 12x_2 = 0 \\ -12x_1 - 9x_2 = 0 \end{cases} \Rightarrow 4x_1 = -3x_2, \quad \text{Let } x_1 = C_1$$

$$\Rightarrow \begin{cases} x_1 = C_1 \\ x_2 = -\frac{4}{3}C_1 \end{cases} \Rightarrow \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ -\frac{4}{3} \end{Bmatrix} \xrightarrow{C_1 = 3} \begin{Bmatrix} 3 \\ -4 \end{Bmatrix}} \quad \text{ANS.}$$

✓ when $\lambda_2 = 25$ $\quad (\underline{A} - \lambda_2 \underline{I})\, \underset{\sim}{u}_2 = \underset{\sim}{0}$

$$\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 9x_1 - 12x_2 = 0 \\ -12x_1 + 16x_2 = 0 \end{cases} \Rightarrow 3x_1 = 4x_2, \quad \text{Let } x_1 = C_2$$

$$\Rightarrow \begin{cases} x_1 = C_2 \\ x_2 = \frac{3}{4}C_2 \end{cases} \Rightarrow \underline{\underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_2 \begin{Bmatrix} 1 \\ \frac{3}{4} \end{Bmatrix} \xrightarrow{C_2 = 4} \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}} \quad \text{ANS.}$$

Normalize eigenvectors:

$$\frac{u_1}{|u_1|} = \frac{1}{\sqrt{3^2+(-4)^2}}\begin{Bmatrix} 3 \\ -4 \end{Bmatrix} = \begin{Bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{Bmatrix}$$

$$\frac{u_2}{|u_2|} = \frac{1}{\sqrt{4^2+3^2}}\begin{Bmatrix} 4 \\ 3 \end{Bmatrix} = \begin{Bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{Bmatrix}$$

Modal Matrix:

$$M = [u_1, u_2] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

ANS.

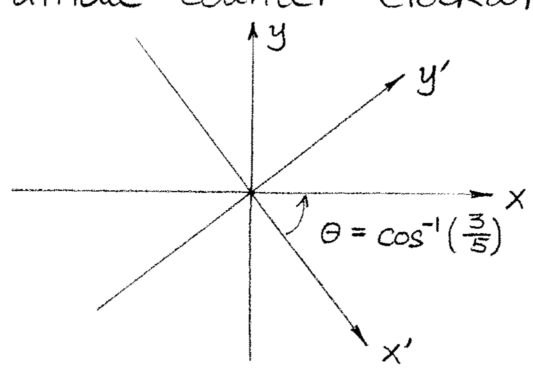
(C) $$\begin{pmatrix} x \\ y \end{pmatrix} \equiv M \begin{pmatrix} x' \\ y' \end{pmatrix}$$

General form for 2D coordinate transform:

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

So: $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$

✓ θ is angle rotates from x'-y' coordinate to x-y coordinate counter-clockwise



(d)

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$M^T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

So: $M^{-1} = M^T$

(e)

$$F \equiv \begin{pmatrix} x \\ y \end{pmatrix}^T A \begin{pmatrix} x \\ y \end{pmatrix} = \left[M\begin{pmatrix} x' \\ y' \end{pmatrix}\right]^T A \left[M\begin{pmatrix} x' \\ y' \end{pmatrix}\right] = C$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}^T M^T A M \begin{pmatrix} x' \\ y' \end{pmatrix} = C$$

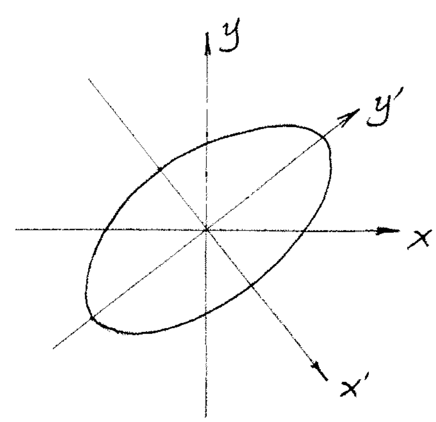
Since $M^{-1} = M^T$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}^T M^{-1} A M \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}^T \begin{bmatrix} 50 & 0 \\ 0 & 25 \end{bmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = C$$

$$\Rightarrow 50x'^2 + 25y'^2 = C$$

$$\Rightarrow \left(\frac{x'}{\sqrt{\frac{C}{50}}}\right)^2 + \left(\frac{y'}{\sqrt{\frac{C}{25}}}\right)^2 = 1$$

Ans.



✓ Major axis:

direction → along y' axis

magnitude → $\sqrt{\frac{C}{25}}$

✓ Minor axis:

direction → along x' axis

magnitude → $\sqrt{\frac{C}{50}}$

Problem 3

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 4 & 3 \\ -1 & -3 & -2 \end{bmatrix}$$

(a) $(\underline{A} - \lambda \underline{I})\underset{\sim}{u} = \underset{\sim}{0}$

$$\det|\underline{A} - \lambda\underline{I}| = \begin{vmatrix} 3-\lambda & 6 & 6 \\ 1 & 4-\lambda & 3 \\ -1 & -3 & -2-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)(-2-\lambda) + 9(3-\lambda)$$

$$= (3-\lambda)(\lambda^2 - 2\lambda + 1) = 0$$

$$\Rightarrow \quad \underline{\lambda_1 = 3, \quad \lambda_2 = \lambda_3 = 1} \quad \text{ANS.}$$

✓ when $\lambda_1 = 3$, $\quad (\underline{A} - \lambda_1 \underline{I})\underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & 1 & 3 \\ -1 & -3 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 6x_2 + 6x_3 = 0 \longrightarrow \boxed{x_2 = -x_3} \\ x_1 + x_2 + 3x_3 = 0 \longrightarrow x_1 + 2x_3 = 0 \\ -x_1 - 3x_2 - 5x_3 = 0 \longrightarrow -x_1 - 2x_3 = 0 \end{cases} \Rightarrow \boxed{x_1 = -2x_3}$$

Let $x_1 = C_1$

$$\Rightarrow \begin{cases} x_1 = C_1 \\ x_2 = \frac{1}{2}C_1 \\ x_3 = -\frac{1}{2}C_1 \end{cases} \Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{Bmatrix} \xrightarrow{C_1 = 2} \begin{Bmatrix} 2 \\ 1 \\ -1 \end{Bmatrix}$$

ANS.

when $\lambda_2 = \lambda_3 = 1 \qquad (\underset{\sim}{A} - \lambda \underset{\sim}{I})\underset{\sim}{u} = \underset{\sim}{0}$

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 3 & 3 \\ -1 & -3 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 + 6x_2 + 6x_3 = 0 \\ x_1 + 3x_2 + 3x_3 = 0 \\ -x_1 - 3x_2 - 3x_3 = 0 \end{cases} \Rightarrow x_1 = -3x_2 - 3x_2$$

Let $x_2 = C_2$, $x_3 = C_3$

$$\Rightarrow \begin{cases} x_1 = -3C_2 - 3C_3 \\ x_2 = C_2 \\ x_3 = C_3 \end{cases} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \underbrace{\begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix}}_{\underset{\sim}{u}_2} C_2 + \underbrace{\begin{Bmatrix} -3 \\ 0 \\ 1 \end{Bmatrix}}_{\underset{\sim}{u}_3} C_3$$

So:

$$\underset{\sim}{u}_2 = \begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix}, \quad \underset{\sim}{u}_3 = \begin{Bmatrix} -3 \\ 0 \\ 1 \end{Bmatrix}$$

(b) Modal Matrix

$$M = [\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3] = \begin{bmatrix} 2 & -3 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix} \leftarrow \text{from Matlab 7.0.1}$$

$$M^{-1}AM = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 3 & 6 & 6 \\ 1 & 4 & 3 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -3 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Jordan form will show up only when there is no enough eigenvectors. In this problem, although the matrix is not symmetric and has repeated eigenvalues, we still can find 3 independent eigenvectors to build modal matrix. So we can diagonalize matrix and Jordan form won't appear. student X is wrong.

Problem 4:

$$A = \begin{bmatrix} 1 & K \\ -K & 2 \end{bmatrix}$$

(a) $(\underline{A} - \lambda \underline{I})\underset{\sim}{u} = \underset{\sim}{0}$

$$\det|\underline{A} - \lambda\underline{I}| = \begin{vmatrix} 1-\lambda & K \\ -K & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 + K^2 = 0$$

Repeated Eigenvalues:

$$\Delta = (-3)^2 - 4(2+K^2) = 0$$

$$\left.\begin{array}{l} \Rightarrow \quad K = \pm\frac{1}{2} \\ K \text{ is positive real number} \end{array}\right\} \Rightarrow \underline{K = \frac{1}{2}}$$ ANS.

(b) $\lambda^2 - 3\lambda + \frac{9}{4} = 0$

$$\Rightarrow (\lambda - \tfrac{3}{2})^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \tfrac{3}{2}$$

when $\lambda_1 = \lambda_2 = 1$ $\quad (\underline{A} - \lambda\underline{I})\underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow -\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 = 0 \quad \Rightarrow \quad x_1 = x_2, \quad \text{Let } x_1 = C_1$$

$$\Rightarrow \begin{cases} x_1 = C_1 \\ x_2 = C_1 \end{cases} \Rightarrow \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \xrightarrow{C_1 = 1} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$ ANS.

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\, \underset{\sim}{u}_2 = \underset{\sim}{u}_1$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow -\frac{1}{2}x_1 + \frac{1}{2}x_2 = 1 \quad \Rightarrow x_1 = x_2 - 2, \quad \text{Let } x_2 = c_2$$

$$\Rightarrow \begin{cases} x_1 = c_2 - 2 \\ x_2 = c_2 \end{cases} \Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_2 - 2 \\ c_2 \end{Bmatrix} \xrightarrow{c_2 = 1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

ANS.

Note: u_2 is NOT an eigenvector

(c) Modal Matrix:

$$M = [\underset{\sim}{u}_1, \underset{\sim}{u}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{1+1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & \frac{3}{2} \end{bmatrix}$$

ANS.

This is a Jordan form.