ME 374, System Dynamics Analysis and Design
Homework 2

Distributed: 4/7/2008, Due: 4/18/2008
(There are 6 problems in this set.)

1. Consider the state equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

with initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \mathbf{x}_0 \tag{2}$$

Determine the response $x_1(t)$ and $x_2(t)$ of the system by using the following two methods.

(a) Determine the eigenvalues and eigenvectors of the state matrix. Derive $x_1(t)$ and $x_2(t)$ via eigenvector expansion.

(b) Determine the state transition matrix using the series expansion

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots \tag{3}$$

to the fifth power. Derive $x_1(t)$ and $x_2(t)$ in terms of infinite series.

2. We have learned that the homogeneous solution of a state equation can be found through two different methods. The first method is to use eigenvalues and eigenvectors. The second method is to use the state transition matrix. Since the solution of the state equation is unique, the state transition matrix and eigenvalues and eigenvectors must be related through

$$\Phi(t) \equiv e^{\mathbf{A}t} = \mathbf{M}\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}\mathbf{M}^{-1} \tag{4}$$

This problem is to prove (4) through use of similarity transformation.

Assume that $\mathbf{A}$ is an $N \times N$ matrix and let $\lambda_1, \dots, \lambda_N$ be N *distinct* eigenvalues of $\mathbf{A}$ with *linearly independent* eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_N$. Let $\mathbf{M}$ be the modal matrix diagonalizing $\mathbf{A}$, i.e.,

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix} \tag{5}$$

Show that

$$\mathbf{M}^{-1}e^{\mathbf{A}t}\mathbf{M} = \mathbf{I} + t\mathbf{M}^{-1}\mathbf{A}\mathbf{M} + \frac{t^2}{2!}\mathbf{M}^{-1}\mathbf{A}^2\mathbf{M} + \frac{t^3}{3!}\mathbf{M}^{-1}\mathbf{A}^3\mathbf{M} + \dots$$

$$= \mathbf{I} + t\mathbf{M}^{-1}\mathbf{A}\mathbf{M} + \frac{t^2}{2!}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M})(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}) + \frac{t^3}{3!}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M})^3 + \ldots$$

$$= \mathbf{I} + t\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix}^2 + \ldots$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_N t} \end{bmatrix} \tag{6}$$

Derive (4) from (6).

3. This is an old exam problem that I gave in Spring Quarter of 1998. Consider the state equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7}$$

   (a) Find the eigenvalues and eigenvectors of the state matrix.

   (b) Is the system stable? Why? Do the responses $x_1(t)$ and $x_2(t)$ oscillate? Why?

   (c) Explain how you would determine the state transition matrix $\Phi(t)$. You don't have to calculate it. It takes too much time.

   (d) Student X solved (7) and found the state transition matrix to be

$$\Phi(t) = \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix} \tag{8}$$

   Based on Student X's solution, please determine $x_1(t)$ and $x_2(t)$ when $x_1(0) = 1$ and $x_2(0) = 1$.

4. This is an old exam problem that I gave in Spring Quarter of 1998. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 34 & -12 \\ 0 & -12 & 41 \end{bmatrix} \tag{9}$$

   which has a pair of repeated roots $\lambda_1 = \lambda_2 = 25$. Because of the repeated roots, student X concludes that the eigenvector expansion

$$\Phi(t) = \mathbf{M}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}\mathbf{M}^{-1} \tag{10}$$

   cannot be used in finding $e^{\mathbf{A}t}$. Is student X correct or not? Why?

5. This is an exam problem I gave in the Spring Quarter of 1999. Consider the state equation and initial condition

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ k & -4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{11}$$

where k is a constant parameter.

(a) When $k = -3$, find the eigenvalues and eigenvector(s) of the state matrix.

(b) Let the state transition matrix of $\mathbf{A}$ be $\Phi(t) \equiv [\phi_1(t), \phi_2(t)]$, where $\phi_1(t)$ and $\phi_2(t)$ are the first and second columns of $\Phi(t)$. Determine the solution of (11) in terms of $\phi_1(t)$ and $\phi_2(t)$.

(c) Determine the range of k so that the system is oscillatory. Is the system stable when it is oscillatory? (Hint: First show that eigenvalues of the system satisfy $\lambda^2 + 4\lambda - k = 0$. The two eigenvalues are

$$\lambda_{1,2} = -2 \pm \sqrt{4+k} \tag{12}$$

Use (12) to judge.)

(d) When $k = -4$, the system has repeated eigenvalues. Student A wants to use

$$\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \ldots \tag{13}$$

to find the state transition matrix $\Phi(t)$. Student B thinks that the series solution (13) only works when the system does not have repeated eigenvalues. Who is right and why?

6. This is an old exam problem of the Spring Quarter 2006. Consider the state equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{14}$$

where x_1 and x_2 are state variables.

(a) Let $\Phi(t)$ be the state transition matrix. Determine $\Phi(t)$ through eigenvector expansion. Hint: A useful formula is the inversion of 2×2 matrix as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{15}$$

(b) The state equation (14) is subjected to the following initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{16}$$

Determine $x_2(t)$ through use of the state transition matrix.

(c) Determine the stability of the system. Will the response of the system oscillate? Explain why.

(d) Approximate the state transition matrix $\Phi(t)$ through use of

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots \tag{17}$$

to the order of t^3.