Problem 1.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}}_{\underline{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{I.C.} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = x_0$$

(a) $(\underline{A} - \lambda \underline{I})\underset{\sim}{u} = \underset{\sim}{0}$

$$\det\begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 16 = 0$$

So: $\underline{\lambda_1 = 4i, \quad \lambda_2 = -4i}$ Ans.

✓ when $\lambda_1 = 4i$, $(\underline{A} - \lambda_1 \underline{I})\underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} -4ix_1 + 4x_2 = 0 \\ -4x_1 - 4ix_2 = 0 \end{cases} \Rightarrow 4ix_1 = 4x_2, \quad \text{Let } x_1 = c_1$$

$$\Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_1 i \end{Bmatrix} \xrightarrow{c_1 = 1} \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ i \end{Bmatrix}}$$ Ans.

✓ when $\lambda_2 = -4i$, $(\underline{A} - \lambda_2 \underline{I})\underset{\sim}{u}_2 = \underset{\sim}{0}$

$$\begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 4ix_1 + 4x_2 = 0 \\ -4x_1 + 4ix_2 = 0 \end{cases} \Rightarrow 4x_1 = 4ix_2, \quad \text{Let } x_1 = c_2$$

$$\Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_2 \\ -c_2 i \end{Bmatrix} \xrightarrow{c_2 = 1} \underline{\underset{\sim}{u}_2 = \begin{Bmatrix} 1 \\ -i \end{Bmatrix}}$$ Ans.

✓ Modal Matrix:

$$M = [\underset{\sim}{u}_1, \underset{\sim}{u}_2] = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{i}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$\underset{\sim}{X}_h(t) = M\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} M^{-1}\underset{\sim}{X}_0$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{4it} & 0 \\ 0 & e^{-4it} \end{bmatrix}\frac{i}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}\underset{\sim}{X}_0$$

$$= \frac{i}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} -ie^{4it} & -e^{4it} \\ -ie^{-4it} & e^{-4it} \end{bmatrix}\underset{\sim}{X}_0$$

$$= \frac{i}{2}\begin{bmatrix} -i(e^{i4t}+e^{-i4t}) & -e^{i4t}+e^{-i4t} \\ e^{i4t}-e^{-i4t} & -i(e^{i4t}+e^{-i4t}) \end{bmatrix}\underset{\sim}{X}_0$$

✓ Recall Euler Formular:

$$e^{i\theta} = \cos\theta + i\sin\theta\,; \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\text{So: } \underset{\sim}{X}_h(t) = \frac{i}{2}\begin{bmatrix} -2i\cos 4t & -2i\sin 4t \\ 2i\sin 4t & -2i\cos 4t \end{bmatrix}\underset{\sim}{X}_0$$

$$= \begin{bmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{bmatrix}\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$

ANS.

(b) State transition matrix $\Phi(t)$

$$\Phi(t) = e^{\underline{A}t} = \underline{M}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix}\underline{M}^{-1}$$

$$e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{\underline{A}^2 t^2}{2!} + \frac{\underline{A}^3 t^3}{3!} + \cdots$$

$$\underline{A} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}, \qquad \underline{A}^2 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = 4^2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underline{A}^3 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\underline{A}^2 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}4^2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 4^3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{A}^4 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\underline{A}^3 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}4^3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 4^4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{A}^5 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\underline{A}^4 = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}4^4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 4^5\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

So: $$e^{\underline{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}4t + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\frac{4^2t^2}{2!} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{4^3t^3}{3!}$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{4^4t^4}{4!} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{4^5t^5}{5!} + \cdots$$

Thus:

$$\underset{\sim}{X}_h(t) = e^{\underline{A}t}\underset{\sim}{X}_0 = \begin{bmatrix} 1 - \frac{4^2t^2}{2!} + \frac{4^4t^4}{4!} - \cdots & 4t - \frac{4^3t^3}{3!} + \frac{4^5t^5}{5!} - \cdots \\ -4t + \frac{4^3t^3}{3!} - \frac{4^5t^5}{5!} + \cdots & 1 - \frac{4^2t^2}{2!} + \frac{4^4t^4}{4!} - \cdots \end{bmatrix}\underset{\sim}{X}_0$$

ANS.

Problem 2

$$e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{\underline{A}^2t^2}{2!} + \frac{\underline{A}^3t^3}{3!} + \cdots$$

$$\Rightarrow \underline{M}^{-1}e^{\underline{A}t}\underline{M} = \underline{M}^{-1}\underline{I}\,\underline{M} + t\underline{M}^{-1}\underline{A}\,\underline{M} + \frac{t^2}{2!}\underline{M}^{-1}\underline{A}^2\underline{M} + \frac{t^3}{3!}\underline{M}^{-1}\underline{A}^3\underline{M} + \cdots$$

$$= \underline{I} + t\underline{M}^{-1}\underline{A}\,\underline{M} + \frac{t^2}{2!}(\underline{M}^{-1}\underline{A}\,\underline{M})(\underline{M}^{-1}\underline{A}\,\underline{M}) + \frac{t^3}{3!}(\underline{M}^{-1}\underline{A}\,\underline{M})^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + t\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

$$+ \frac{t^2}{2!}\begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N^2 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \lambda_2^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N^3 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2\lambda_1^2}{2!} + \frac{t^3\lambda_1^3}{3!} + \cdots & 0 & \cdots & 0 \\ 0 & 1 + t\lambda_1 + \frac{t^2\lambda_1^2}{2!} + \frac{t^3\lambda_1^3}{3!} + \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + t\lambda_1 + \frac{t^2\lambda_1^2}{2!} + \frac{t^3\lambda_1^3}{3!} + \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_N t} \end{bmatrix}$$

So:

$$\Phi(t) \equiv e^{\underline{A}t} = \underline{M}(\underline{M}^{-1}e^{\underline{A}t}\underline{M})\underline{M}^{-1} = M\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_N t} \end{bmatrix} M^{-1}$$

ANS.

Problem 3:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

(a) $(\underset{\sim}{A} - \lambda \underset{\sim}{I})\, \underset{\sim}{u} = \underset{\sim}{0}$

$$\det\begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = 0$$

$\Rightarrow (\lambda + 1)(\lambda + 6) = 0$

So: $\underline{\lambda_1 = -1}$, $\underline{\lambda_2 = -6}$ ANS.

✓ when $\lambda_1 = -1$ $\quad (\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\, \underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ -6x_1 - 6x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2, \text{ Let } x_1 = C_1$$

$$\Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ -C_1 \end{Bmatrix} \xrightarrow{C_1 = 1} \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}$$ ANS.

✓ when $\lambda_2 = -6$ $\quad (\underset{\sim}{A} - \lambda_2 \underset{\sim}{I})\, \underset{\sim}{u}_2 = \underset{\sim}{0}$

$$\begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

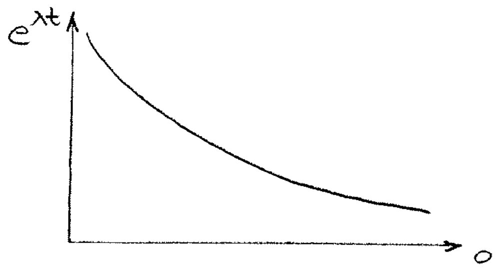
$$\Rightarrow \begin{cases} 6x_1 + x_2 = 0 \\ -6x_1 - x_2 = 0 \end{cases} \Rightarrow 6x_1 = -x_2, \text{ Let } x_1 = C_2$$

$$\Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_2 \\ -6C_2 \end{Bmatrix} \xrightarrow{C_2 = 1} \underset{\sim}{u}_2 = \begin{Bmatrix} 1 \\ -6 \end{Bmatrix}$$

ANS.

(b) $\lambda_1 = -1 < 0$; $\lambda_2 = -6 < 0$

Both eigenvalues are negative real numbers, so system is stable.



System response will NOT oscillate, since two eigenvalues are both real numbers, only imaginary part contains "cos" & "sin" terms.

(c) State transition matrix $\Phi(t)$

$$\underline{M} = [\underset{\sim}{u}_1, \underset{\sim}{u}_2] = \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix}$$ then find $\underline{M}^{-1}$

$$\Phi(t) = \underline{M} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \underline{M}^{-1}$$

(d)

$$\Phi(t) = \begin{bmatrix} e^{t} + e^{-t} & e^{t} - e^{-t} \\ e^{t} - e^{-t} & e^{t} + e^{-t} \end{bmatrix}$$

So:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \Phi(t) \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$

$$= \begin{bmatrix} e^{t} + e^{-t} & e^{t} - e^{-t} \\ e^{t} - e^{-t} & e^{t} + e^{-t} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2e^{t} \\ 2e^{t} \end{Bmatrix}$$

ANS.

Problem 4:

$$\underline{A} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 34 & -12 \\ 0 & -12 & 41 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 25$$

$$(\underline{A} - \lambda I)\, \underset{\sim}{u} = \underset{\sim}{0}$$

$$\det \begin{vmatrix} 25-\lambda & 0 & 0 \\ 0 & 34-\lambda & -12 \\ 0 & -12 & 41-\lambda \end{vmatrix} = (\lambda - 25)^2 (\lambda - 50) = 0$$

So: $\lambda_1 = \lambda_2 = 25$; $\lambda_3 = 50$ ANS.

Student X is NOT correct. Although for this case we have two repeated eigenvalues, matrix A is symmetrix, so we are able to find enough eigenvectors.

So equation: $\Phi(t) = M \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} M^{-1}$

can be used in finding $e^{\underline{A}t}$.

Problem 5:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ K & -4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \qquad \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

(a) $K = -3$

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\,\underset{\sim}{u} = \underset{\sim}{0} \qquad \underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

$$\det\begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

$$\Rightarrow (\lambda + 1)(\lambda + 3) = 0$$

So: $\underline{\lambda_1 = -1, \quad \lambda_2 = -3}$ ANS.

✓ when $\lambda_1 = -1$ $\quad (\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\,\underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ -3x_1 - 3x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2, \quad \text{Let } x_1 = c_1$$

$$\Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ -c_2 \end{Bmatrix} \xrightarrow{c_1 = 1} \underline{\underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}$$ ANS.

✓ when $\lambda_2 = -3$ $\quad (\underset{\sim}{A} - \lambda_2 \underset{\sim}{I})\,\underset{\sim}{u}_2 = \underset{\sim}{0}$

$$\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 3x_1 + x_2 = 0 \\ -3x_1 - x_2 = 0 \end{cases} \Rightarrow 3x_1 = -x_2, \quad \text{Let } x_1 = C_2$$

$$\Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_2 \\ -3C_2 \end{Bmatrix} \xrightarrow{C_2 = 1} \underline{\underset{\sim}{u}_2 = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}}$$ ANS.

(b)

$$\underset{\sim}{X}_h(t) = \Phi(t)\, \underset{\sim}{X}_0$$

$$= \left[\, \underset{\sim}{\phi}_1(t),\ \underset{\sim}{\phi}_2(t) \,\right] \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

$$= \underline{\underset{\sim}{\phi}_1(t) - 2\, \underset{\sim}{\phi}_2(t)}$$ ANS.

(c)

$$(\underline{A} - \lambda \underline{I})\, \underset{\sim}{u} = \underset{\sim}{0} \qquad \underline{A} = \begin{bmatrix} 0 & 1 \\ K & -4 \end{bmatrix}$$

$$\det \begin{vmatrix} -\lambda & 1 \\ K & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - K = 0$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 + 4K}}{2} = -2 \pm \sqrt{4 + K}$$

✓ System is oscillatory, eigenvalue must be two complex conjugates.

$$4 + K < 0 \quad \Rightarrow \quad \underline{K < -4}$$ ANS.

System is stable!

(d) Student A is correct. The series solution

$$e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{\underline{A}^2 t^2}{2!} + \cdots$$ will work all the time, and it does NOT depend on existence of repeated eigenvalues.

Problem 6:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

(a) $(\underline{A} - \lambda \underline{I})\, \underset{\sim}{u} = \underset{\sim}{0}$

$$\det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

So: $\lambda_1 = 1$, $\lambda_2 = -1$

✓ when $\lambda_1 = 1$ $\quad (\underline{A} - \lambda_1 \underline{I})\, \underset{\sim}{u}_1 = \underset{\sim}{0}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \Rightarrow x_1 = x_2, \quad \text{Let } x_1 = c_1$$

$$\Rightarrow \underset{\sim}{u}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_1 \end{Bmatrix} \xrightarrow{c_1 = 1} \underset{\sim}{u}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

✓ when $\lambda_2 = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2, \quad \text{Let } x_1 = c_2$$

$$\Rightarrow \underset{\sim}{u}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_2 \\ -c_2 \end{Bmatrix} \xrightarrow{c_2=1} \underset{\sim}{u}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

So:

$$\underline{M} = [\underset{\sim}{u}_1, \underset{\sim}{u}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{M}^{-1} = \frac{1}{\det M} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

State transition matrix $\Phi(t)$

$$\Phi(t) = M \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -e^t & -e^t \\ -e^{-t} & e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}$$

$$= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

ANS.

(b) I.C.

$$\underset{\sim}{X}(0) = \begin{Bmatrix} 2 \\ -3 \end{Bmatrix}$$

$$\underset{\sim}{X}_h(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \Phi(t) \cdot \underset{\sim}{X}(0) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \begin{Bmatrix} 2 \\ -3 \end{Bmatrix}$$

$$\Rightarrow \underline{X_2(t) = 2\cosh t - 3\sinh t}$$ ANS.

(c) The system is NOT stable, because $Re[\lambda_1] > 0$.

The response will NOT oscillate, because there are NO imaginary part in λ_1 & λ_2.

(d) $$\Phi(t) = e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \underline{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\underline{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e^{\underline{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 + \frac{t^2}{2!} + \cdots & t + \frac{t^3}{3!} + \cdots \\ t + \frac{t^3}{3!} + \cdots & 1 + \frac{t^2}{2!} + \cdots \end{bmatrix}$$

ANS.