Problem 1:

$$RC\dot{x} + x = RC\dot{u}(t)$$

(a) $x(t) = X(s)e^{st}, \quad u(t) = U(s)e^{st}$

$$\Rightarrow RCsX(s)\cancel{e^{st}} + X(s)\cancel{e^{st}} = RCsU(s)\cancel{e^{st}}$$

$$\Rightarrow (RCs+1)X(s) = RCsU(s)$$

$$\Rightarrow H(s) = \frac{X(s)}{U(s)} = \frac{RCs}{RCs+1}$$ ANS.

(b) when $R = C = 1$, $u(t) = \sin t$

$u(t) = \sin t = \mathrm{Im}[e^{jt}]$, So $U(s) = 1$, $s = j$

$$H(s) = \frac{RCs}{RCs+1} = \frac{j}{j+1} = \frac{j(j-1)}{(j+1)(j-1)} = \frac{1}{2} + \frac{1}{2}j$$

So:

$$|H(s)| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$ ANS.

$$\measuredangle H(s) = \tan^{-1}\left(\frac{1/2}{1/2}\right) = \frac{\pi}{4}$$ ANS.

(c) when $R = C = 1$, $u(t) = \sin t$

$u(t) = \sin t = \mathrm{Im}[e^{jt}]$, So $U(s) = 1$, $s = j$

$$X(s) = H(s)U(s) = \left(\frac{1}{2} + \frac{1}{2}j\right) \times 1 = \frac{1}{2} + \frac{1}{2}j$$

$$x(t) = \left[X(s)e^{st}\right]_{s=j} = \left(\frac{1}{2} + \frac{1}{2}j\right)e^{jt} = \frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}e^{jt} = \frac{\sqrt{2}}{2}e^{j(\frac{\pi}{4}+t)}$$

$$|x(t)| = \frac{\sqrt{2}}{2}, \qquad \measuredangle x(t) = \frac{\pi}{4} + t$$ ANS.

(d) $u(t) = \cos\omega t = Re[e^{j\omega t}]$

So: $U(S) = 1$, $S = j\omega$

$$H(S) = \frac{RCS}{RCS+1} = \frac{RC\omega j}{RC\omega j + 1}$$

$$\measuredangle H(S) = \measuredangle[RC\omega j] - \measuredangle[RC\omega j + 1]$$

$$= \frac{\pi}{2} - \tan^{-1}(RC\omega) = \frac{\pi}{4}$$

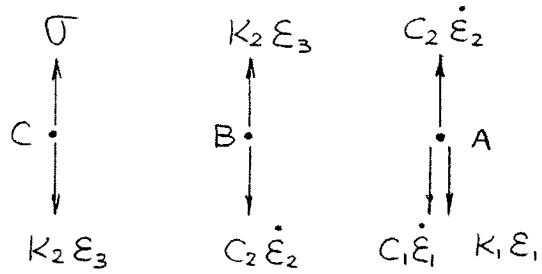
$$\Rightarrow \tan^{-1}(RC\omega) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

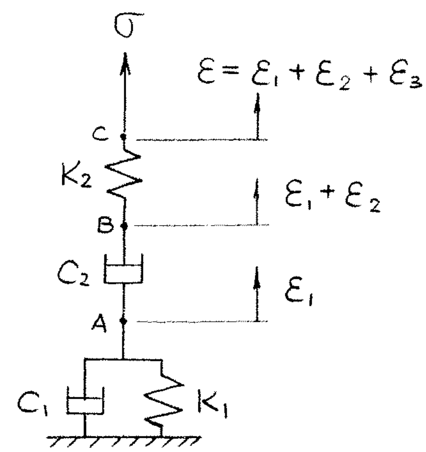
$$\Rightarrow RC\omega = 1$$

$$\Rightarrow \omega = \frac{1}{RC}$$ ANS.

Problem 2:

(a) FBD




Point C: $\Sigma F_c$: $\sigma - K_2 \varepsilon_3 = 0 \Rightarrow \sigma = K_2 \varepsilon_3$ ①

Point B: $\Sigma F_B$: $K_2 \varepsilon_3 - C_2 \dot{\varepsilon}_2 = 0 \Rightarrow \sigma = C_2 \dot{\varepsilon}_2$ ②

Point A: $\Sigma F_A$: $C_2 \dot{\varepsilon}_2 - (C_1 \dot{\varepsilon}_1 + K_1 \varepsilon_1) = 0 \Rightarrow \sigma = C_1 \dot{\varepsilon}_1 + K_1 \varepsilon_1$ ③

Since: $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

So: $\Rightarrow \varepsilon_1 = \varepsilon - \varepsilon_2 - \varepsilon_3$

Eqn ③: $\sigma = C_1 \dot{\varepsilon}_1 + K_1 \varepsilon_1$

$$= C_1 (\dot{\varepsilon} - \dot{\varepsilon}_2 - \dot{\varepsilon}_3) + K_1 (\varepsilon - \varepsilon_2 - \varepsilon_3)$$

thus: $\dot{\sigma} = C_1 (\ddot{\varepsilon} - \ddot{\varepsilon}_2 - \ddot{\varepsilon}_3) + K_1 (\dot{\varepsilon} - \dot{\varepsilon}_2 - \dot{\varepsilon}_3)$ ④

Eqn ①: $\varepsilon_3 = \dfrac{\sigma}{K_2} \Rightarrow \dot{\varepsilon}_3 = \dfrac{\dot{\sigma}}{K_2} \Rightarrow \ddot{\varepsilon}_3 = \dfrac{\ddot{\sigma}}{K_2}$

Eqn ②: $\dot{\varepsilon}_2 = \dfrac{\sigma}{C_2} \Rightarrow \ddot{\varepsilon}_2 = \dfrac{\dot{\sigma}}{C_2}$

So, Eqn ④: $\dot{\sigma} = C_1 \left(\ddot{\varepsilon} - \dfrac{\dot{\sigma}}{C_2} - \dfrac{\ddot{\sigma}}{K_2}\right) + K_1 \left(\dot{\varepsilon} - \dfrac{\sigma}{C_2} - \dfrac{\dot{\sigma}}{K_2}\right)$

$$\Rightarrow \frac{K_1}{C_2}\sigma + \left(1 + \frac{K_1}{K_2} + \frac{C_1}{C_2}\right)\dot{\sigma} + \frac{C_1}{K_2}\ddot{\sigma} = K_1 \dot{\varepsilon} + C_1 \ddot{\varepsilon}$$ ⑤

ANS.

(b) $\varepsilon(t) = \varepsilon_0(s)e^{st}$ , $\sigma(t) = \sigma_0(s)e^{st}$

Eqn ⑤ : $\frac{K_1}{C_2}\sigma_0(s)\cancel{e^{st}} + \left(1 + \frac{K_1}{K_2} + \frac{C_1}{C_2}\right)\sigma_0(s)\cdot s\cancel{e^{st}} + \frac{C_1}{K_2}\sigma_0(s)\cdot s^2\cancel{e^{st}}$

$$= K_1\varepsilon_0(s)\cdot s\cancel{e^{st}} + C_1\varepsilon_0(s)s^2\cancel{e^{st}}$$

$$\Rightarrow \left\{\frac{K_1}{C_2} + \left(1 + \frac{K_1}{K_2} + \frac{C_1}{C_2}\right)s + \frac{C_1}{K_2}s^2\right\}\sigma_0(s) = \left(K_1 s + C_1 s^2\right)\varepsilon_0(s)$$

So : Transfer Function

$$H(s) = \frac{\sigma_0(s)}{\varepsilon_0(s)} = \frac{K_1 s + C_1 s^2}{\frac{K_1}{C_2} + \left(1 + \frac{K_1}{K_2} + \frac{C_1}{C_2}\right)s + \frac{C_1}{K_2}s^2}$$

ANS.

(c) $C_1 = C_2 = 1$ , $K_1 = K_2 = 10$

$$H(s) = \frac{10s + s^2}{10 + (1+1+1)s + 0.1s^2} = \frac{s^2 + 10s}{0.1s^2 + 3s + 10}$$

input : $\bar{\varepsilon}$ (constant)

$$\bar{\varepsilon} = \varepsilon_0(s)e^{st} \Rightarrow \varepsilon_0(s) = \bar{\varepsilon}, \quad s = 0$$

So : $H(s) = \left.\frac{s^2 + 10s}{0.1s^2 + 3s + 10}\right|_{s=0} = 0$ , $\angle H(s)|_{s=0} = 0$

$$\bar{\sigma} = \bar{\varepsilon}\, H(s) = 0$$

ANS.

steady-state stress is zero.

(d) input: $\varepsilon(t) = \cos 10t = Re\left[e^{j10t}\right]$

$$\Rightarrow \quad \mathcal{E}_o(S) = 1, \quad S = 10j, \quad \measuredangle \varepsilon(t) = 10t$$

$$\text{So: } H(S) = \frac{(10j)^2 + 10(10j)}{0.1(10j)^2 + 3(10j) + 10} = \frac{-100 + 100j}{-\cancel{10} + 30j + \cancel{10}}$$

$$= \frac{-100+100j}{30j} = \frac{(-100+100j)j}{30j \cdot j} = \frac{10}{3}(1+j)$$

$$|H(S)| = \frac{10}{3} \times \sqrt{2} = \frac{10\sqrt{2}}{3}, \quad \measuredangle H(S) = \tan^{-1}(1) = \frac{\pi}{4}$$

output $\sigma(t)$:

$$\sigma_o(S) = H(S) \cdot \mathcal{E}_o(S) = \frac{10}{3}(1+j) \times 1 = \frac{10}{3}(1+j)$$

$$\sigma(t) = \left[\sigma_o(s) e^{st}\right]_{s=10j} = \frac{10}{3}(1+j) e^{j10t}$$

$$= \frac{10\sqrt{2}}{3} e^{j(10t + \frac{\pi}{4})}$$

ANS.

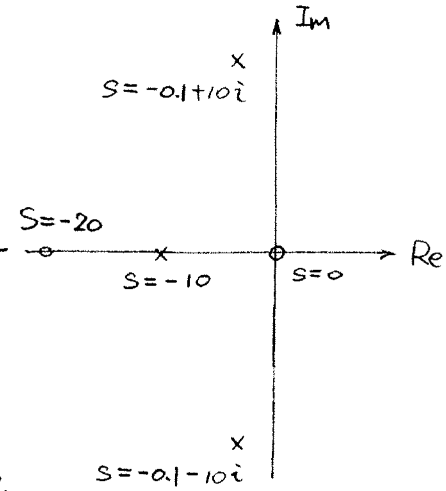
## Problem 3:

(a) Transfer Function:

$$H(s) = \frac{(s-0)[s-(-20)]}{[s-(-10)][s-(-0.1+10i)][s-(-0.1-10i)]}$$

$$= \frac{s(s+20)}{(s+10)(s^2+0.2s+100.01)}$$

Ans.




(b) ODE:

$$(s+10)(s^2+0.2s+100.01) = 0$$

order of ODE is highest order of s.

So: 3rd order Ans.

$$\Rightarrow (s+10)(s+0.1-10i)(s+0.1+10i) = 0$$

$$\Rightarrow \begin{cases} s_1 = -10 \\ s_2 = -0.1+10i \\ s_3 = -0.1-10i \end{cases}$$

Ans.

(c) input: $\cos\omega t = Re[e^{j\omega t}] \Rightarrow s = j\omega$

✓ To get a minimal response, we need to let "s" move close to or right at "zero", $s=j\omega$, thus, when $\omega=0$, $s=0$, we are right at zero, so system has a minimized response.

✓ To get a maximal response, just let "s" move close to or right at one of the poles. When $\omega=10$ ($\omega$ is frequency, it can NOT be negative), s is close to the pole.

Problem 4:

Governing Eqn: $\frac{3}{2} m\ddot{x} + c\dot{x} + Kx = c\dot{u} + Ku$

(a) $x(t) = X(s)e^{st}$ , $u(t) = U(s)e^{st}$

$$\Rightarrow \left(\frac{3}{2}mS^2 + cS + K\right) X(s)\cancel{e^{st}} = (cS + K)U(s)\cancel{e^{st}}$$

$$\Rightarrow H(s) = \frac{X(s)}{U(s)} = \frac{cS + K}{\frac{3}{2}mS^2 + cS + K}$$

ANS.

(b) input $u(t) = e^{-2t} \Rightarrow U(s) = 1$ , $S = -2$

$$H(s) = \left.\frac{cS + K}{\frac{3}{2}mS^2 + cS + K}\right|_{S=-2} = \frac{-2C + K}{6m - 2C + K}$$

$$X(s) = H(s)U(s) = \frac{-2C + K}{6m - 2C + K}$$

$$\Rightarrow x(t) = \left[X(s)e^{st}\right]_{S=-2} = \frac{-2C + K}{6m - 2C + K} e^{-2t}$$

ANS.

(c) input $u(t) = e^{-2t} \Rightarrow U(s) = 1$ , $S = -2$

$$H(s) = \frac{-2C + K}{6m - 2C + K}$$

$$|H(s)| = \frac{-2C + K}{6m - 2C + K} , \quad \measuredangle H(s) = \tan^{-1}(0) = 0$$

ANS.

(d) zero:

$$Cs + K = 0 \quad \Rightarrow \quad s_1 = -\frac{K}{C}$$

pole:

$$\frac{3}{2} m s^2 + Cs + K = 0$$

$$s_{1,2} = \frac{-C \pm \sqrt{C^2 - 6mK}}{3m}$$

Case I: $C^2 > 6mK$

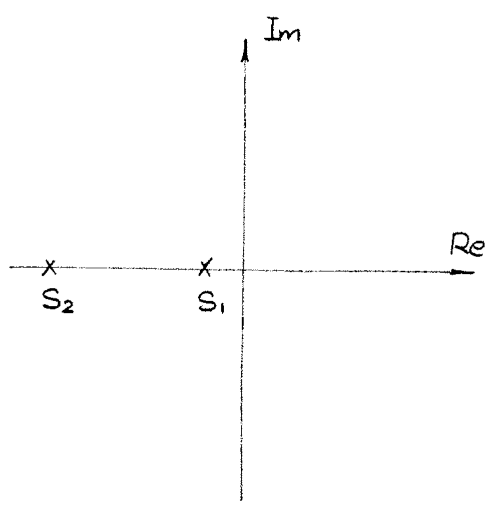
$$s_{1,2} = -\frac{C}{3m} \pm \frac{1}{3m}\sqrt{C^2 - 6mK}$$

Case II: $C^2 < 6mK$

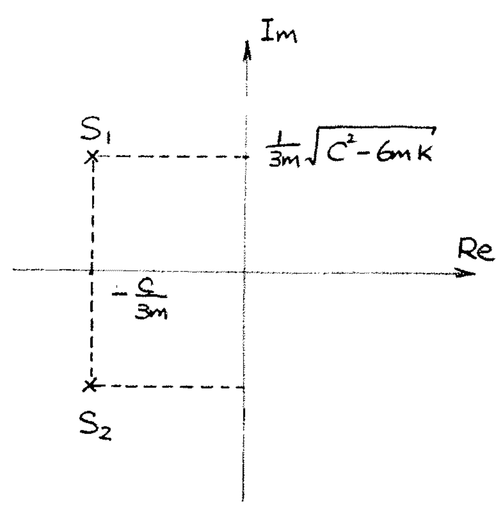
$$s_{1,2} = -\frac{C}{3m} \pm \frac{1}{3m}\sqrt{C^2 - 6mK}$$

ANS.

(e) Case I:




Case II:



## Problem 5:

$$\ddot{x} + 0.1\dot{x} + (10^6 + K)x = \dot{u}(t)$$

(a) $x(t) = X(S)e^{st}$, $u(t) = U(S)e^{st}$

$$\Rightarrow \left\{ S^2 + 0.1S + (10^6 + K) \right\} X(S)\cancel{e^{st}} = S\,U(S)\cancel{e^{st}}$$

$$\Rightarrow H(S) = \frac{X(S)}{U(S)} = \frac{S}{S^2 + 0.1S + (10^6+K)}$$ ANS.

(b) $K = 2\times 10^6$, $u(t) = \cos 1000t = Re[\,e^{j1000t}\,]$

$$\Rightarrow U(S) = 1, \quad S = j1000$$

$$H(S) = \frac{1000j}{(1000j)^2 + 0.1\times 1000j + (10^6 + 2\times 10^6)} = 2.5\times 10^{-8} + 0.5\times 10^{-3}j$$

$$|H(S)| = \sqrt{(2.5\times 10^{-8})^2 + (0.5\times 10^{-3})^2} = 5\times 10^{-4}$$ ANS.

$$\measuredangle H(S) = \tan^{-1}\left(\frac{0.5\times 10^{-3}}{2.5\times 10^{-8}}\right) = 89.9971^\circ = 1.5707 \text{ rad}$$ ANS.

(C) Avoid shaking means avoid big response.

$$H(S) = \frac{S}{S^2 + 0.1S + (10^6+K)}$$

pole:

$$D(S) = S^2 + 0.1S + (10^6 + K) = 0$$

$$\Rightarrow S_{1,2} = \frac{-0.1 \pm \sqrt{0.01 - 4(10^6+K)}}{2}$$

$$= -0.05 \pm \sqrt{0.0025 - (10^6+K)} \approx -0.05 \pm j\sqrt{10^6+K}$$

$$\omega = \sqrt{10^6 + K} = 6280 \text{ rad/s}$$

$$\Rightarrow \underline{K = 3.84 \times 10^7} \text{ ANS.}$$

If K avoids to be set at $3.84 \times 10^7$, we can avoid system poles, $-0.05 \pm 6280j$, thus avoid violent shake.

(d) $K = 2 \times 10^6$, designed operating frequency is 2200 rad/s

$$H(s) = \frac{s}{s^2 + 0.1s + (10^6 + 2\times10^6)} = \frac{s}{s^2 + 0.1s + 3\times10^6}$$

Pole:

$$D(s) = s^2 + 0.1s + 3\times10^6 = 0$$

$$\Rightarrow s_{1,2} = \frac{-0.1 \pm \sqrt{0.01 - 4\times3\times10^6}}{2}$$

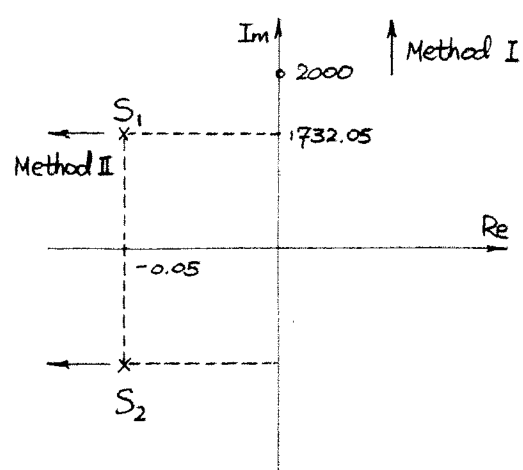
$$= -0.05 \pm 1732.05j$$

System damped natural frequency: $\omega = 1732.05$ rad/s

Designed operating frequency: $\Omega = 2000$ rad/s

Method I: Increase operating frequency, let $\Omega$ move away from pole.

Method II: Increase system damping, let pole move away from operating frequency.




Are you on the right track so far?... QING@ME B114, Apr 21st 20