ME 374, System Dynamics Analysis and Design
Homework 4

Distributed: 4/21/2008, Due: 5/2/2008

(There are 4 problems in this set.)

1. Consider a mass-damper system shown in Fig. 1 with inertia m and damping coefficient c. The mass is driven by a prescribed displacement $u(t)$. In addition, the position of the mass is $x(t)$ and the velocity of the mass is $v(t) \equiv \dot{x}(t)$.

   (a) Show that the equation of motion governing $v(t)$ is

   $$m\dot{v} + cv = c\dot{u} \tag{1}$$

   Find the transfer function from $u(t)$ to $v(t)$.

   (b) Determine the poles and zeros in terms of m and c. What is the physical driving condition at the zero?

   (c) From now on, consider the following parameters $m = 1$ kg and $c = 1$ Ns/m. When $u(t) = \sin 2t$, calculate the transfer function and determine $v(t)$.

   (d) Consider the following input

   $$u(t) = \cos \omega t \tag{2}$$

   where ω is an arbitrary driving frequency. Derive the magnitude and phase of the transfer function as a function of ω. What is the frequency ω such that the phase of the transfer function is 45°? In addition, what is the magnitude of the transfer function at that frequency?

   (e) Now consider the equation of motion governing $x(t)$. Show that it satisfies

   $$m\ddot{x} + c\dot{x} = c\dot{u} \tag{3}$$

   Derive the transfer function from $u(t)$ to $x(t)$. What are the poles and zeros of this transfer function? Is there a pole-zero cancellation? If there is, what is the physical meaning of the pole-zero cancellation.

2. This is a real problem that we encounter in our hard disk drive (HDD) vibration research. In the past, HDD used ball bearings generating a lot of noise and vibration. Currently, most HDD on the market adopt hydrodynamic bearings. Figure 2 shows a simplified model. The inner circle in Fig. 2 represents the shaft carrying all the disks, the outer circle in Fig. 2 represents the bearing sleeve, and the shaded area is the radial hydrodynamic bearing. Let's define a coordinate system xy with its origin attached to the center of the bearing sleeve. The motion of the shaft is then described by the coordinates x and y of the shaft center. The vibration of HDD spindles with hydrodynamic bearings is very complicated, but a simplified version of the equations of motion takes the following form.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_x(t) \\ f_y(t) \end{pmatrix} \tag{4}$$

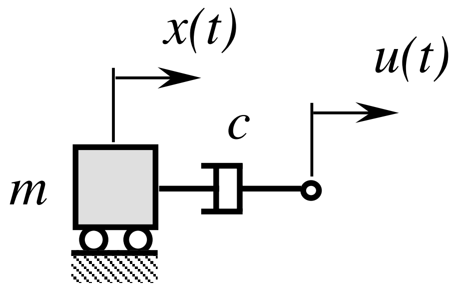

Figure 1: A mass-damper system

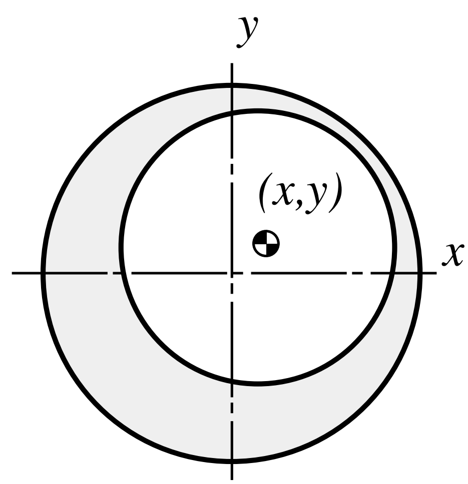

Figure 2: Whirling of a spindle with hydrodynamic bearing

In (4), m is the mass of the spindle (including the disks) and c is the damping coefficient of the bearing. In addition, k_1 and k_2 are called in-line stiffness and cross stiffness of a hydrodynamic bearing. f_x and f_y are the forces acting on the spindle (e.g., vibration from your laptop computer). To make our analysis simple, assume that $f_y = 0$.

(a) Show that the transfer function from f_x to x is

$$H_x(s) = \frac{ms^2 + cs + k_1}{(ms^2 + cs + k_1)^2 + k_2^2} \tag{5}$$

and the transfer function from f_x to y is

$$H_y(s) = \frac{k_2}{(ms^2 + cs + k_1)^2 + k_2^2} \tag{6}$$

Hint: Assume that

$$f_x(t) = F(s)e^{st}, \quad x(t) = X(s)e^{st}, \quad y(t) = Y(s)e^{st} \tag{7}$$

Substitute (7) into (4), and H_x and H_y can be obtained from

$$H_x(s) = \frac{X(s)}{F(s)}, \quad H_y(s) = \frac{Y(s)}{F(s)} \tag{8}$$

(b) Show that the systems have four poles. Two of them are governed by

$$ms^2 + cs + k_1 - jk_2 = 0 \tag{9}$$

and the other two are the complex conjugate of the poles from (9)

(c) Now let's use the parameters that appear in a real disk drive. $m = 7.872 \times 10^{-2}$, $c = 4.158 \times 10^4$, $k_1 = 1.727 \times 10^4 \omega_3$, and $k_2 = 2.185 \times 10^4 \omega_3$, where ω_3 is the spin speed of the disk drive in rad/s. (Note that k_1 and k_2 are proportional to the spin speed ω_3.) Calculate and plot the poles of the system on the complex plan for $\omega_3 = 5,400$ rpm and $\omega_3 = 7,200$ rpm.

(d) Consider an exciation

$$f_x(t) = \cos \omega t \tag{10}$$

where ω is the excitation frequency. Find out the frequency ω at which the spindle will have the maximum response. (Hint: For the input in (10), $s = j\omega$. When will s get closest to the poles?) You should find that the excitation frequency ω causing the maximum response is always very close to the half of the spin speed ω_3. Therefore, this phenomenon is oftern called *half-speed whirl.*

(e) Consider the case when $\omega_3 = 7,200$ rpm and the excitation frequency $\omega = 125$ Hz. Calculate the magnitude and phase of $H_x(s)$ and $H_y(s)$. Will the spindle have a larger response in x or in y? Why?

(f) Consider the case when $\omega_3 = 7,200$ rpm and the excitation frequency $\omega = 110$ Hz. Calculate $x(t)$ and $y(t)$. Plot the spindle positions (x, y) on the cartesian coordinates as a function of time. What do you see?

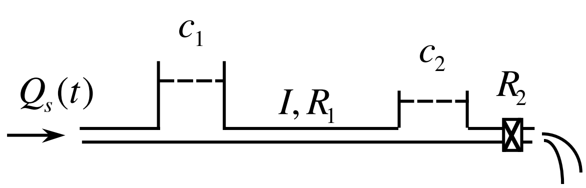

Figure 3: Another fluid system

3. This problem requires some calculations. You can use your fancy calculator or any software to work out the answers. I have listed some commands in Matlab, in case you need them.

Consider the fluid system shown in Fig. 3. Let $C_1 = C_2 = I = R_1 = R_2 = 1$. The state equation and output equation are

$$\frac{d}{dt}\begin{pmatrix} p_{C_1} \\ p_{C_2} \\ q_I \end{pmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{pmatrix} p_{C_1} \\ p_{C_2} \\ q_I \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} Q_s \tag{11}$$

and

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} p_{C_1} \\ p_{C_2} \\ q_I \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} Q_s \tag{12}$$

(a) Determine the stability of the system by evaluating the eigenvalues of the state matrix **A**. (Hint: Use Matlab command eig(**A**).)

(b) Judging from the eigenvalues, there should be two modes of water discharge from the tank. Describe qualitatively how the water levels in the tank will vary as functions of time for each mode of discharge.

(c) Determine the transfer matrix from the input Q_s to the output pressures p_1 and p_2. (Hint: Use the formula $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.)

(d) Look at $H_1(s)$, which is the transfer function from Q_s to p_1. $H_1(s)$ should take the form of

$$H_1(s) = \frac{s^2 + 2s + 2}{s^3 + 2s^2 + 3s + 1} \tag{13}$$

The poles of the system then satisfy

$$s^3 + 2s^2 + 3s + 1 = 0 \tag{14}$$

Find the roots of the polynomial (i.e., the poles) by using Matlab command roots($\mathbf{C}$), where $\mathbf{C}$ is a vector containing the coefficients of the polynomial in the decending order, i.e.,

$$\mathbf{C} = [1 \quad 2 \quad 3 \quad 1] \tag{15}$$

Determine the zeros in a similar manner. Plot the poles and zeros on the complex s plane.

4. This is an old exam problem from the Spring Quarter of 2006. Consider the state equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \tag{16}$$

where x_1 and x_2 are state variables, and $u(t)$ is the input variable.

(a) Derive the transfer function $H_1(s)$ from $u(t)$ to $x_1(t)$.

(b) Determine the poles and zeros of $H_1(s)$.