ME 374, System Dynamics Analysis and Design
Homework 4: Solution (May 6, 2008)
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Problem 1

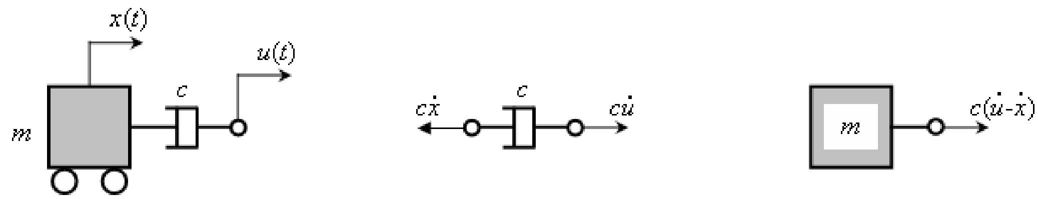


Figure 1: Mass-damper system; free-body diagrams

(a) From the free body diagram, the sum of the forces on the mass m is

$$\Sigma F_x = c(\dot{u} - \dot{x}) = m\ddot{x}.$$

Since $v(t) \equiv \dot{x}(t)$,

$$c(\dot{u} - v) = m\dot{v},$$

or

$$\underline{\underline{m\dot{v} + cv = c\dot{u}}}. \quad (Ans.)$$

Substuting $v(t) = V(s)e^{st}$ and $u(t) = U(s)e^{st}$ gives

$$msX(s)e^{st} + cX(s)e^{st} = csU(s)e^{st}$$

or

$$(ms + c)\, X(s) = csU(s).$$

Then, the transfer function from $u(t)$ to $x(t)$ is

$$\underline{\underline{H(s) = \frac{X(s)}{U(s)} = \frac{cs}{ms + c}}}. \quad (Ans.)$$

(b) **pole:** $s = \frac{-c}{m}$; **zero:** $s = 0$.

At the zero $s = 0$,

$$u(t) = U(s)e^{st}\Big|_{s=0} = U(0) \cdot (1) = U(0).$$

Therefore, the *physical driving condition at the zero* is that $\underline{\underline{u(t) = constant}}$.

(c) With $m = 1$kg and $c = 1$Ns/m,

$$H(s) = \frac{s}{s+1}.$$

Then, $u(t) = \sin 2t = \text{Im}\{e^{2jt}\} \Rightarrow s = 2j$ $(U(s) = 1)$. Substituting into $H(s)$ gives

$$H(2j) = \frac{2j}{1+2j} \cdot \frac{1-2j}{1-2j} = \frac{4+2j}{5}.$$

Since the input $u(t)$ is the imaginary component of a complex expression, the output $v(t)$ will also be the imaginary component, i.e.,

$$v(t) = \text{Im}\left\{V(s)e^{st}\right\}\Big|_{s=2j} = \text{Im}\left\{H(2j)U(2j)e^{2jt}\right\} = \text{Im}\left\{\frac{4+2j}{5}(\cos 2t + j\sin 2t)\right\},$$

or

$$\underline{\underline{v(t) = \frac{2}{5}\cos 2t + \frac{4}{5}\sin 2t.}} \quad (Ans.)$$

(d) Now, given $u(t) = \cos\omega t = \text{Re}\{e^{j\omega t}\} \Rightarrow s = j\omega$, $(U(s) = 1)$. Then, $H(s)$ becomes

$$H(s)\big|_{s=j\omega} = \frac{s}{s+1}\Big|_{s=j\omega} = \frac{j\omega}{1+j\omega}\cdot\frac{1-j\omega}{1-j\omega} = \frac{\omega^2 - j\omega}{1+\omega^2}$$

$$H(s)\big|_{s=j\omega} = \frac{\omega}{1+\omega^2}(\omega + j).$$

$$\textbf{Magnitude}: \ /H(s)/\big|_{s=j\omega} = \sqrt{(\text{Re}\{H(s)\})^2 + (\text{Im}\{H(s)\})^2}\Big|_{s=j\omega}$$

$$\underline{\underline{/H(s)/\big|_{s=j\omega} = \frac{\omega}{1+\omega^2}\sqrt{\omega^2+1}.}} \quad (Ans.)$$

$$\textbf{Phase}: \angle H(s)\big|_{s=j\omega} = \tan^{-1}\left(\frac{\text{Im}\{H(s)\}}{\text{Re}\{H(s)\}}\right)$$

$$\underline{\underline{\angle H(s)\big|_{s=j\omega} = \tan^{-1}\left(\frac{1}{\omega}\right).}} \quad (Ans.)$$

For $\angle H(s) = 45° = \pi/4$,

$$\tan^{-1}\left(\frac{1}{\omega}\right) = \frac{\pi}{4}$$

$$\frac{1}{\omega} = 1$$

or

$$\underline{\underline{\omega = 1 \text{ rad/s.}}} \quad (Ans.)$$

Then,

$$/H(s)/_{\omega=1} = \frac{1}{1+1^2}\sqrt{1^2+1}$$

$$\underline{\underline{/H(s)/_{\omega=1} = \frac{\sqrt{2}}{2}.}} \quad (Ans.)$$

(e) Substituting $x(t) = X(s)e^{st}$ and $u(t) = U(s)e^{st}$ into $m\ddot{x} + c\dot{x} = c\dot{u}$ gives

$$(ms^2 + cs)X(s) = csU(s).$$

Then,

$$\underline{\underline{H(s) = \frac{X(s)}{U(s)} = \frac{cs}{ms^2+cs}.}} \quad (Ans.)$$

zero: $s = 0$;   **poles:** $s = 0$, $s = \frac{-c}{m}$.

There is pole-zero cancellation at $s = 0$. Then, we might think the transfer function looks like

$$H(s) = \frac{X(s)}{U(s)} = \frac{c}{ms + c},$$

but this does not represent the equation of motion $m\ddot{x} + c\dot{x} = c\dot{u}$ governing $x(t)$. If the system is excited at $s = 0$ (i.e. a constant force is applied), then the position of the mass described by $x(t)$ cannot be determined.

Problem 2

(a) Substituting $x(t) = X(s)e^{st}$, $y(t) = Y(s)e^{st}$, and $f_x(t) = F(s)e^{st}$ into the equations of motion

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(t) \\ 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} s^2X(s)e^{st} \\ s^2Y(s)e^{st} \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} sX(s)e^{st} \\ sY(s)e^{st} \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix} \begin{bmatrix} X(s)e^{st} \\ Y(s)e^{st} \end{bmatrix} = \begin{bmatrix} F(s)e^{st} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} ms^2 + cs + k_1 & k_2 \\ -k_2 & ms^2 + cs + k_1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} ms^2 + cs + k_1 & k_2 \\ -k_2 & ms^2 + cs + k_1 \end{bmatrix}^{-1} \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{(ms^2 + cs + k_1)^2 + k_2^2} \begin{bmatrix} ms^2 + cs + k_1 & -k_2 \\ k_2 & ms^2 + cs + k_1 \end{bmatrix} \begin{bmatrix} F(s) \\ 0 \end{bmatrix}.$$

Therefore,

$$X(s) = \frac{ms^2 + cs + k_1}{(ms^2 + cs + k_1)^2 + k_2^2} F(s)$$

$$Y(s) = \frac{k_2}{(ms^2 + cs + k_1)^2 + k_2^2} F(s),$$

so

$$\underline{\underline{H_x(s) = \frac{X(s)}{F(s)} = \frac{ms^2 + cs + k_1}{(ms^2 + cs + k_1)^2 + k_2^2}}}, \quad \underline{\underline{H_y(s) = \frac{Y(s)}{F(s)} = \frac{k_2}{(ms^2 + cs + k_1)^2 + k_2^2}}}. \quad (Ans.)$$

(b) The denominator is the same for $H_x(s)$ and $H_y(s)$, and is a fourth-order polynomial in s. Therefore, $H_x(s)$ and $H_y(s)$ have four poles, which are found from

$$(ms^2 + cs + k_1)^2 + k_2^2 = 0$$

or

$$(ms^2 + cs + k_1)^2 = -k_2^2.$$

Taking the square root of both sides (since complex numbers are permitted) gives

$$ms^2 + cs + k_1 = \pm jk_2.$$

Then the poles are determined from

$$ms^2 + cs + k_1 - jk_2 = 0 \quad (1)$$

$$ms^2 + cs + k_1 + jk_2 = 0. \quad (2)$$

The poles from (1) are the complex conjugates of the poles from (2).

(c) We are given parameters $m = 7.872 \times 10^{-2}$, $c = 4.158 \times 10^4$, $k_1 = 1.727 \times 10^4 \omega_3$, $k_2 = 2.185 \times 10^4 \omega_3$, where ω_3 is in rad/s.

For $\omega_3 = 5,400$ rpm $= 565.4867$ rad/s, the poles of (1) and (2) are

$$\underline{\underline{\begin{aligned} p_1 &= -5.2797 \times 10^5 - 297.42j, \quad p_2 = -234.81 + 297.42j \\ p_3 &= -5.2797 \times 10^5 + 297.42j, \quad p_4 = -234.81 - 297.42j. \end{aligned}}} \quad (Ans.)$$

Note that p_1 and p_3 are complex conjugates, as are p_2 and p_4.

For $\omega_3 = 7,200$ rpm $= 753.9822$ rad/s, the poles of (1) and (2) are

$$\underline{\underline{\begin{aligned} p_1 &= -5.2789 \times 10^5 - 396.68j, \quad p_2 = -313.05 + 396.68j \\ p_3 &= -5.2789 \times 10^5 + 396.68j, \quad p_4 = -313.05 - 396.68j. \end{aligned}}} \quad (Ans.)$$

Again, note that p_1 and p_3 are complex conjugates, as are p_2 and p_4.

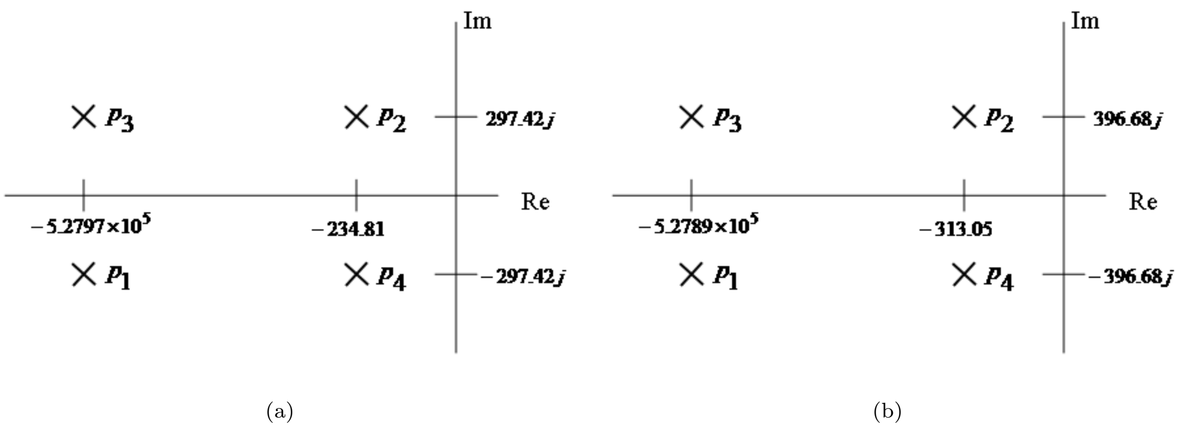


Figure 2: Poles: (a) $\omega_3 = 5,400$ rpm; (b) $\omega_3 = 7,200$ rpm

(d) The excitation $f_x(t) = \cos \omega t = \text{Re}\{e^{j\omega t}\} \Rightarrow s = j\omega$. The output of a system experiences the maximum response when the system is excited at the poles of the transfer function. From (c), all four poles are complex with negative real parts. The maximum response occurs at the poles closest to the imaginary axis (i.e. p_2 and p_4). Since $s = j\omega$ from the input excitation, the maximum response occurs at $\omega = \text{Im}\{p_2\}$.

At $\omega_3 = 5,400$ rpm $= 565.4867$ rad/s,

$$\underline{\underline{\omega = \text{Im}\{p_2\} = 297.42 \text{ rad/s} \approx \frac{1}{2}\omega_3.}} \quad (Ans.)$$

At $\omega_3 = 7,200$ rpm $= 753.9822$ rad/s,

$$\underline{\underline{\omega = \text{Im}\{p_2\} = 396.68 \text{ rad/s} \approx \frac{1}{2}\omega_3.}} \quad (Ans.)$$

(e) The excitation $f_x(t) = \cos\omega t = \text{Re}\{e^{j\omega t}\} \Rightarrow s = j\omega$ $(F(s) = 1)$, with $\omega = 125$ Hz $= 250\pi$ rad/s and $\omega_3 = 7,200$ rpm $= 753.9822$ rad/s. Then, with the same values for m, c, k_1, and k_2 as in (c),

$$\begin{aligned} H_x(s) &= \frac{ms^2 + cs + k_1}{(ms^2 + cs + k_1)^2 + k_2^2} \\ &= \frac{(1.2973 + 3.2657j) \times 10^7}{(-6.2677 + 8.4730j) \times 10^{14}} \\ &= (1.7591 - 2.8323j) \times 10^{-8}, \end{aligned}$$

so

$$\underline{\underline{/H_x(s)/ = 3.3341 \times 10^{-8}}}. \quad (Ans.)$$

Likewise,

$$\begin{aligned} H_y(s) &= \frac{k_2}{(ms^2 + cs + k_1)^2 + k_2^2} \\ &= \frac{1.6475 \times 10^7}{(-6.2677 + 8.4730j) \times 10^{14}} \\ &= (-0.9296 - 1.2567j) \times 10^{-8}, \end{aligned}$$

and

$$\underline{\underline{/H_y(s)/ = 1.5632 \times 10^{-8}}}. \quad (Ans.)$$

Note that the magnitude of $H_x(s)$ is larger than the magnitude of $H_y(s)$. The steady-state response of $x(t)$ and $y(t)$ can be determined by

$$\begin{aligned} x(t) &= \text{Re}\{X(s)e^{st}\} = \text{Re}\{H_x(s)F(s)e^{st}\} = \text{Re}\{H_x(s)e^{st}\} \\ y(t) &= \text{Re}\{Y(s)e^{st}\} = \text{Re}\{H_y(s)F(s)e^{st}\} = \text{Re}\{H_y(s)e^{st}\}. \end{aligned}$$

Since $s = j\omega$, the magnitude of $H_x(s)$ will define the magnitude (or amplitude) of $x(t)$, and the magnitude of $H_y(s)$ will define the magnitude of $y(t)$. Since the magnitude of $H_x(s)$ is larger than the magnitude of $H_y(s)$, the spindle will have a larger response in x.

(f) With $\omega = 110$ Hz $= 220\pi$ rad/s and $\omega_3 = 7,200$ rpm $= 753.9822$ rad/s, and with the same values for m, c, k_1, and k_2 as in (c),

$$\begin{aligned} H_x(s) &= \frac{ms^2 + cs + k_1}{(ms^2 + cs + k_1)^2 + k_2^2} \\ &= \frac{(1.2984 + 2.8738j) \times 10^7}{(-3.8589 + 7.4625j) \times 10^{14}} \\ &= (2.3286 - 2.9440j) \times 10^{-8}, \end{aligned}$$

and

$$\begin{aligned} H_y(s) &= \frac{k_2}{(ms^2 + cs + k_1)^2 + k_2^2} \\ &= \frac{1.6475 \times 10^7}{(-3.8589 + 7.4625j) \times 10^{14}} \\ &= (-0.9007 - 1.7419j) \times 10^{-8}. \end{aligned}$$

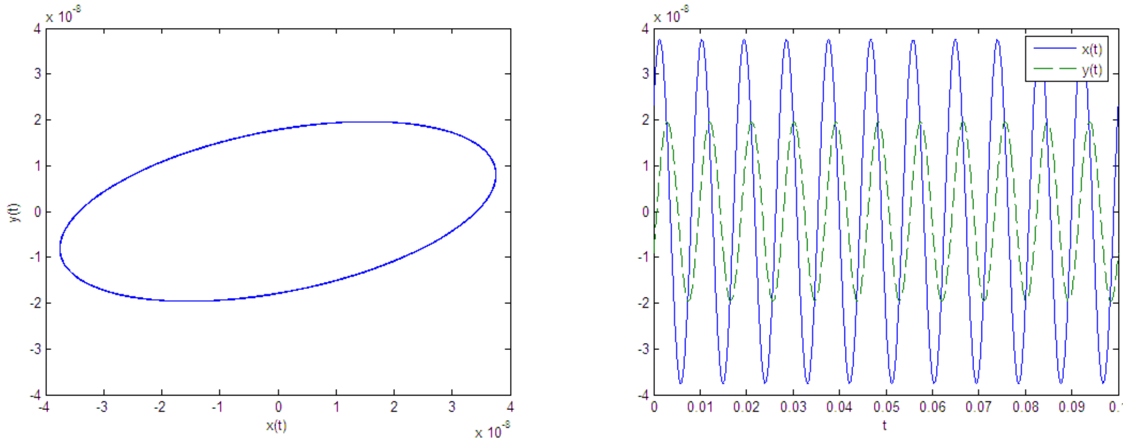
Then,

$$\begin{aligned} &x(t) = \text{Re}\{H_x(s)e^{st}\} = \text{Re}\{(2.3286 - 2.9440j) \times 10^{-8} \cdot (\cos(220\pi t) + j\sin(220\pi t))\} \\ &\underline{\underline{x(t) = 2.3286 \times 10^{-8}\cos(220\pi t) + 2.9440 \times 10^{-8}\sin(220\pi t)}}, \quad (Ans.) \end{aligned}$$

and

$$y(t) = \text{Re}\{H_y(s)e^{st}\} = \text{Re}\{(-0.9007 - 1.7419j) \times 10^{-8} \cdot (\cos(220\pi t) + j\sin(220\pi t))\}$$
$$\underline{\underline{y(t) = -0.9007 \times 10^{-8}\cos(220\pi t) + 1.7419 \times 10^{-8}\sin(220\pi t)}}. \quad (Ans.)$$

From the plot of $x(t)$ vs. $y(t)$, the spindle experiences more displacement in x than in y.



(a) (b)

Figure 3: Spindle position: (a) $x(t)$ vs. $y(t)$; (b) $x(t), y(t)$ vs. t

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Problem 3

We are given the system equations

$$\frac{d}{dt}\begin{bmatrix} p_{C_1} \\ p_{C_2} \\ q_I \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} p_{C_1} \\ p_{C_2} \\ q_I \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} Q_s$$
$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} p_{C_1} \\ p_{C_2} \\ q_I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} Q_s,$$

which looks like

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}u.$$

(a) The eigenvalues of $\mathbf{A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ can be found using the MATLAB command "eig($\mathbf{A}$)", which gives

$$\underline{\underline{\lambda_1 = -0.4302, \quad \lambda_2 = -0.7849 + 1.307i, \quad \lambda_3 = -0.7849 - 1.307i}}. \quad (Ans.)$$

For all three eigenvalues, $\text{Re}\{\lambda\} < 0$. Therefore, the system is $\underline{\underline{\text{stable}}}$.

(b) Two modes of discharge:

- when $\lambda = \lambda_1 = -0.4302$, the water levels in the tank will decrease exponentially.

- when $\lambda = \lambda_{2,3} = -0.7849 \pm 1.307i$, the water levels in the tank will oscillate while decreasing exponentially.

(c) The transfer matrix $\mathbf{H}(s)$ from the input Q_s to the output pressures p_1 and p_2 looks like

$$\mathbf{H}(s) = \begin{bmatrix} H_1(s) \\ H_2(s) \end{bmatrix} = \begin{bmatrix} P_1(s)/\bar{Q}_s(s) \\ P_2(s)/\bar{Q}_s(s) \end{bmatrix},$$

if we let $p_1(t) = P_1(s)e^{st}$, $p_2(t) = P_2(s)e^{st}$, and $Q_s(t) = \bar{Q}_s(s)e^{st}$.

The transfer matrix can be found using $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$. First, note that

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & 0 & 1 \\ 0 & s+1 & -1 \\ -1 & 1 & s+1 \end{bmatrix}^{-1} = \frac{1}{s^3 + 2s^2 + 3s + 1} \begin{bmatrix} s^2 + 2s + 2 & 1 & -(s+1) \\ 1 & s^2 + s + 1 & s \\ s+1 & -s & s^2 + s \end{bmatrix}.$$

Then,

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ &= \frac{1}{s^3 + 2s^2 + 3s + 1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 + 2s + 2 & 1 & -(s+1) \\ 1 & s^2 + s + 1 & s \\ s+1 & -s & s^2 + s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{s^3 + 2s^2 + 3s + 1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 + 2s + 2 \\ 1 \\ s+1 \end{bmatrix} \\ \underline{\underline{\mathbf{H}(s) = \frac{1}{s^3 + 2s^2 + 3s + 1} \begin{bmatrix} s^2 + 2s + 2 \\ 1 \end{bmatrix}}}. &\quad (Ans.) \end{aligned}$$

(d) From (c), the transfer function from Q_s to p_1 is

$$H_1(s) = \frac{s^2 + 2s + 2}{s^3 + 2s^2 + 3s + 1},$$

which has poles satisfying $s^3 + 2s^2 + 3s + 1 = 0$. Using the MATLAB command "roots($\mathbf{R}$)", where $\mathbf{R}$ is a vector containing the coefficients of the polynomial (i.e. $\mathbf{R} = [1\ 2\ 3\ 1]$), the poles are

$$\underline{\underline{s = -0.4302, \quad s = -0.7849 + 1.307i, \quad s = -0.7849 - 1.307i}} \quad (Ans.)$$

(which are the eigenvalues of $\mathbf{A}$).

Similarly, the zeros of $H_1(s)$ satisfy $s^2 + 2s + 2 = 0$. Using the MATLAB command "roots($\mathbf{T}$)", where $\mathbf{T} = [1\ 2\ 2]$, the zeros are

$$\underline{\underline{s = -1 + i, \quad s = -1 - i.}} \quad (Ans.)$$

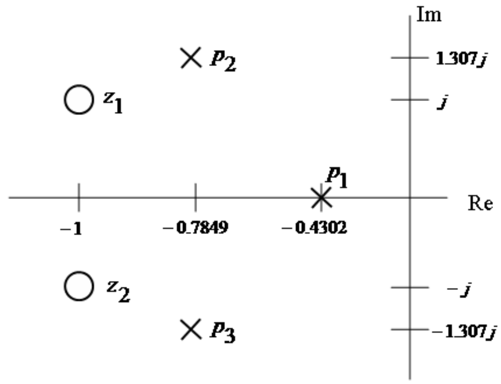

Figure 4: Fluid system poles and zeros

Problem 4

The state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

(a) To find the transfer function $H_1(s)$, first let $x_1(t) = X_1(s)e^{st}$, $x_2(t) = X_2(s)e^{st}$, and $u(t) = U(s)e^{st}$, and substitute into the state equation, which gives

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s)$$

$$\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s).$$

This can be solved via

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s)$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{s^2-1} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s).$$

Then,

$$X_1(s) = \frac{s}{s^2-1} U(s).$$

Therefore, the transfer function from $u(t)$ to $x_1(t)$ is

$$\underline{\underline{H_1(s) = \frac{X_1(s)}{U(s)} = \frac{s}{s^2-1}}}. \quad (Ans.)$$

(b) For $H_1(s)$

- **zero**: $s = 0$;
- **poles**: $s^2 - 1 = 0 \Rightarrow s = \pm 1$.