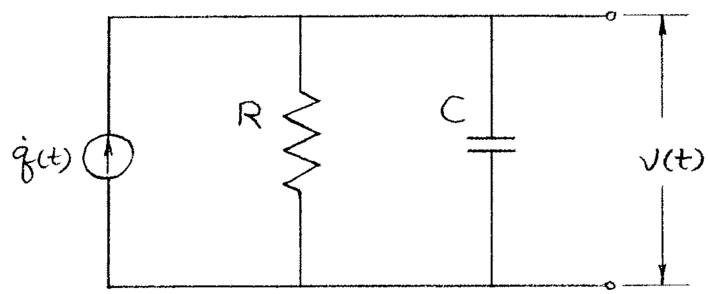
Problem 1:



$$RC\frac{dv}{dt} + v = R\frac{dq}{dt}$$

(a) $\quad v(t) = V(s)e^{st}, \quad q(t) = Q(s)e^{st}$

$$\Rightarrow RC\,V(s)\,s\cancel{e^{st}} + V(s)\cancel{e^{st}} = R\,Q(s)\,s\cancel{e^{st}}$$

$$\Rightarrow (RCs + 1)\,V(s) = Rs\,Q(s)$$

$$\Rightarrow H(s) = \frac{V(s)}{Q(s)} = \frac{Rs}{RCs + 1}$$

FRF:

$$G(\omega) = H(j\omega) = \frac{R\omega j}{RC\omega j + 1} = \frac{R\omega j}{RC\omega j + 1} \cdot \frac{1 - RC\omega j}{1 - RC\omega j}$$

$$= \boxed{\frac{R^2C\omega^2}{1 + R^2C^2\omega^2} + \frac{R\omega}{1 + R^2C^2\omega^2}\,j}$$ ANS.

(b) $$|G(\omega)| = \sqrt{\left(\frac{R^2C\omega^2}{1 + R^2C^2\omega^2}\right)^2 + \left(\frac{R\omega}{1 + R^2C^2\omega^2}\right)^2}$$

$$= \frac{R\omega}{1 + R^2C^2\omega^2}\sqrt{R^2C^2\omega^2 + 1} = \frac{R\omega}{\sqrt{R^2C^2\omega^2 + 1}}$$ ANS

(c) when $\omega \ll 1/RC$

$$|G(\omega)| = \frac{R\omega}{\sqrt{R^2C^2\omega^2+1}} = \frac{\omega/C}{\sqrt{\omega^2+1/R^2C^2}} \quad (\underline{1/R^2C^2 \text{ will dominate}})$$

$$\approx \frac{\omega/C}{\sqrt{1/R^2C^2}} = \omega R$$

ANS

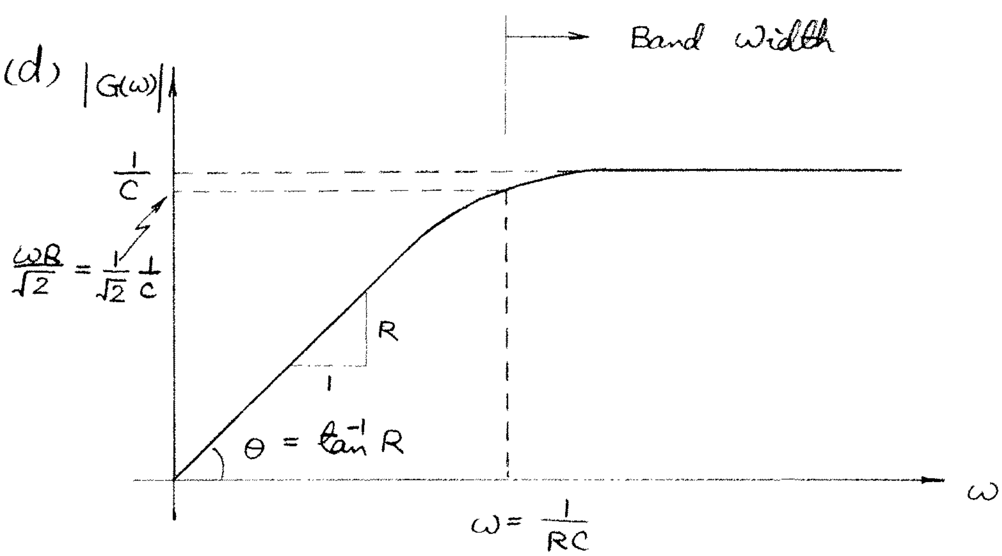
when $\omega \gg 1/RC$

$$|G(\omega)| = \frac{R\omega}{\sqrt{R^2C^2\omega^2+1}} = \frac{\omega/C}{\sqrt{\omega^2+1/R^2C^2}} \quad (\omega \text{ will dominate})$$

$$\approx \frac{\omega/C}{\sqrt{\omega^2}} = \frac{1}{C}$$

ANS

(d)



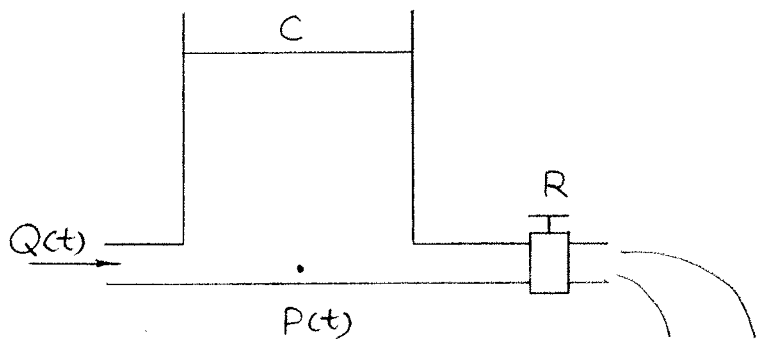
ANS

(e) This is a High - Pass filter,

The frequency range we can use is: $\boxed{\omega > \frac{1}{RC}}$

ANS

Problem 2:



(a) $RC\frac{dP(t)}{dt} + P(t) = RU(t)$

$P(t) = P(s)e^{st}$, $U(t) = U(s)e^{st}$

$\Rightarrow RCsP(s)\cancel{e^{st}} + P(s)\cancel{e^{st}} = RU(s)\cancel{e^{st}}$

$\Rightarrow (RCs + 1)P(s) = RU(s)$

$\Rightarrow H(s) = \frac{P(s)}{U(s)} = \frac{R}{RCs + 1}$

FRF:

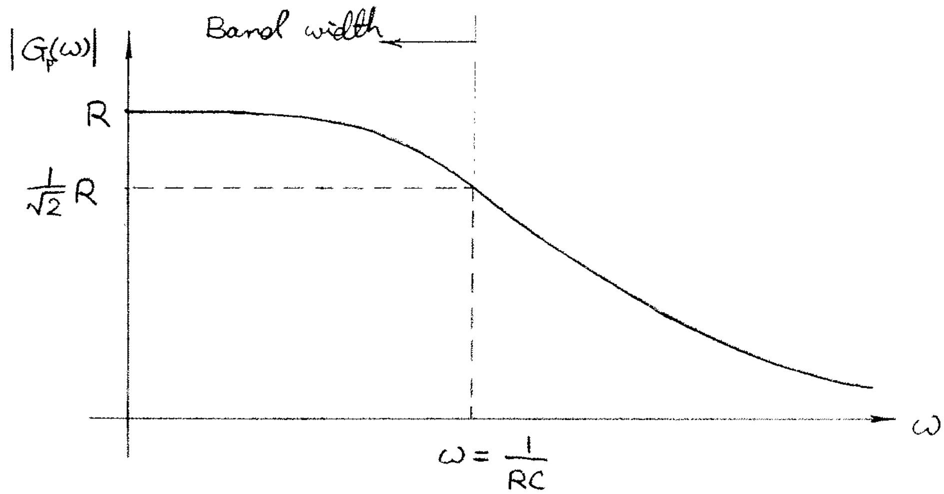
$G_p(s) = H(j\omega) = \frac{R}{RC\omega j + 1} = \frac{R}{RC\omega j + 1} \cdot \frac{1 - RC\omega j}{1 - RC\omega j}$

$$= \frac{R}{1+R^2C^2\omega^2} - \frac{R^2C\omega}{1+R^2C^2\omega^2} j$$

(b)

$$|G_p(\omega)| = \sqrt{\left(\frac{R}{1+R^2C^2\omega^2}\right)^2 + \left(-\frac{R^2C\omega}{1+R^2C^2\omega^2}\right)^2}$$

$$= \frac{R}{1+R^2C^2\omega^2}\sqrt{1+R^2C^2\omega^2} = \frac{R}{\sqrt{1+R^2C^2\omega^2}}$$



when $\omega \ll \frac{1}{RC}$

$$|G_p(\omega)| = \frac{R}{\sqrt{1+R^2C^2\omega^2}} = \frac{\cancel{R}/\cancel{R}C}{\sqrt{1/R^2C^2+\omega^2}} \quad \left(\frac{1}{R^2C^2} \text{ will dominate}\right)$$

$$\approx \frac{1/C}{\sqrt{1/R^2C^2}} = R$$

ANS

when $\omega \gg \frac{1}{RC}$

$$|G_p(\omega)| = \frac{R}{\sqrt{1+R^2C^2\omega^2}} = \frac{\cancel{R}/\cancel{R}C}{\sqrt{1/R^2C^2+\omega^2}} \quad (\omega^2 \text{ will dominate})$$

$$\approx \frac{1/C}{\sqrt{\omega^2}} = \frac{1}{C\omega}$$

ANS

This is a Low-Pass filter.

when $\boxed{\omega \gg \frac{1}{RC}}$, the tank pressure $p(t)$ does NOT fluctuate significantly.

(C)

$$G_c(\omega) = \frac{jRC\omega}{1+jRC\omega} = \frac{jRC\omega}{1+jRC\omega} \cdot \frac{1-jRC\omega}{1-jRC\omega}$$

$$= \frac{R^2C^2\omega^2}{1+R^2C^2\omega^2} + \frac{RC\omega}{1+R^2C^2\omega^2} j$$

$$|G_c(\omega)| = \sqrt{\left(\frac{R^2C^2\omega^2}{1+R^2C^2\omega^2}\right)^2 + \left(\frac{RC\omega}{1+R^2C^2\omega^2}\right)^2}$$

$$= \frac{RC\omega}{1+R^2C^2\omega^2}\sqrt{R^2C^2\omega^2+1}$$

$$= \frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}}$$

ANS

when $\omega \ll \frac{1}{RC}$

$$|G_c(\omega)| = \frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}} = \frac{\omega}{\sqrt{1/R^2C^2+\omega^2}}$$

$$\approx \frac{\omega}{\sqrt{1/R^2C^2}} = RC\omega$$

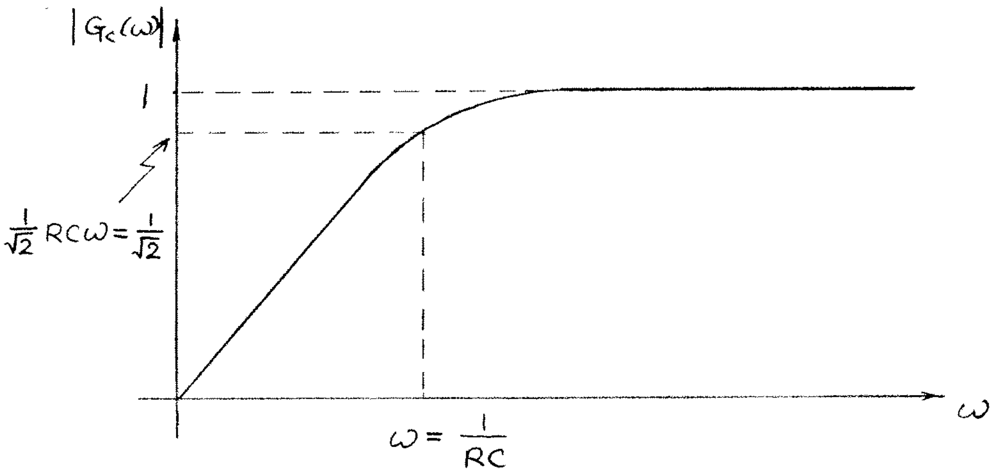
ANS

when $\omega \gg \frac{1}{RC}$

$$|G_c(\omega)| = \frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}} = \frac{\omega}{\sqrt{1/R^2C^2+\omega^2}}$$

$$\approx \frac{\omega}{\sqrt{\omega^2}} = 1$$

ANS



This is a High-Pass filter.

(d)

$$|G(\omega)| = \frac{|Q_c(S)|}{|U(S)|} < \frac{A/2}{A} = \frac{1}{2}$$

$$\Rightarrow \left.\frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}}\right|_{\omega = 200\,Hz} < \frac{1}{2}$$

$$\omega = 2\pi f = \underline{2\pi \times 200} \text{ rad/s}$$

$$\Rightarrow \frac{R^2C^2\omega^2}{1 + R^2C^2\omega^2} < \frac{1}{4}$$

$$\Rightarrow 4R^2C^2\omega^2 < 1 + R^2C^2\omega^2$$

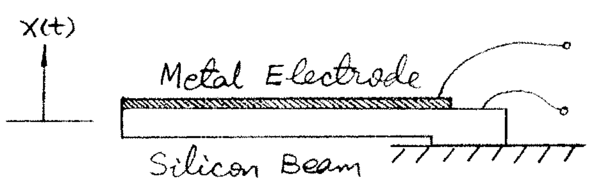
$$\Rightarrow 3R^2C^2\omega^2 < 1$$

$$\Rightarrow R^2C^2 < \frac{1}{3\omega^2}$$

$$\Rightarrow RC < \left.\frac{1}{\sqrt{3\omega^2}}\right|_{\omega = 2\pi \times 200 \text{ rad/s}} = 4.59 \times 10^{-4}$$

ANS

Problem 3:



$$V(t) \rightarrow \boxed{R} \xrightarrow{q(t)} \boxed{G(\omega)} \xrightarrow{T(t)} \boxed{K} \xrightarrow{X(t)}$$

(a)

$$\rho V C \frac{dT(t)}{dt} + hAT(t) = q(t)$$

$$T(t) = T(s)e^{st}, \quad q(t) = Q(s)e^{st}$$

$$\Rightarrow \rho V c s T(s) \cancel{e^{st}} + h A T(s) \cancel{e^{st}} = Q(s) \cancel{e^{st}}$$

$$\Rightarrow (\rho V c s + h A) T(s) = Q(s)$$

$$\Rightarrow H(s) = \frac{T(s)}{Q(s)} = \frac{1}{\rho V c s + h A}$$

<u>FRF</u>:

$$G(\omega) = H(j\omega) = \frac{1}{\rho V c \omega j + hA}$$

$$= \frac{1}{\rho V c \omega j + hA} \cdot \frac{hA - \rho V c \omega j}{hA - \rho V c \omega j}$$

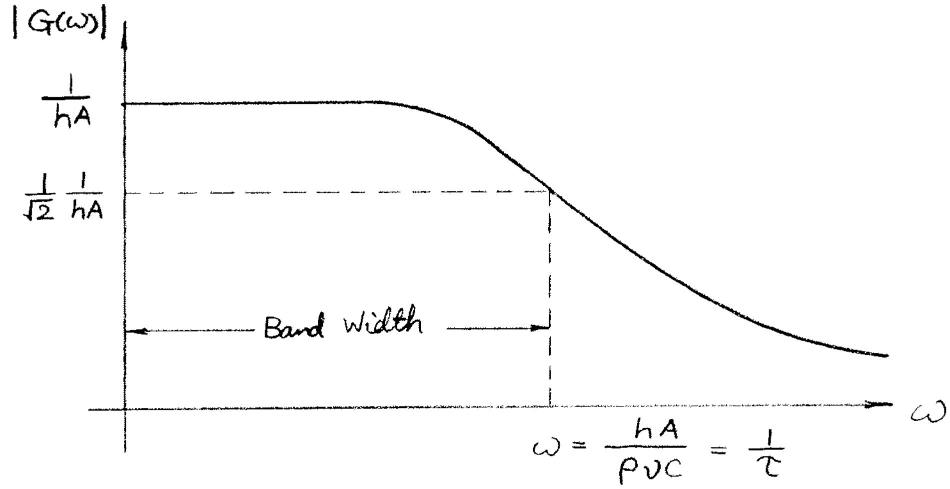
$$= \frac{hA}{h^2 A^2 + \rho^2 V^2 c^2 \omega^2} - \frac{\rho V c \omega}{h^2 A^2 + \rho^2 V^2 c^2 \omega^2} j$$

(b) $$|G(\omega)| = \sqrt{\left(\frac{hA}{h^2 A^2 + \rho^2 V^2 c^2 \omega^2}\right)^2 + \left(-\frac{\rho V c \omega}{h^2 A^2 + \rho^2 V^2 c^2 \omega^2}\right)^2}$$

$$= \frac{1}{\sqrt{h^2 A^2 + \rho^2 V^2 c^2 \omega^2}} = \frac{1}{hA} \cdot \frac{1}{\sqrt{1 + \frac{\rho^2 V^2 c^2}{h^2 A^2} \omega^2}}$$

when $\omega \ll \frac{hA}{\rho V c}$, $|G(\omega)| \approx \frac{1}{hA}$

when $\omega \gg \frac{hA}{\rho V c}$, $|G(\omega)| \approx \frac{1}{\rho V c \omega}$



This is a Low-Pass filter.

Band width: $0 < \omega < \frac{1}{\tau}$

Time constant: $\tau = \frac{\rho V c}{hA}$

ANS

(C) same density: $\rho_A = \rho_B$

volume of Beam: $V = \underset{\text{Thickness}}{\overset{\text{Surface Area}}{AH}} \rightarrow \begin{cases} A_A = 2A_B \\ H_A = 2H_B \end{cases} \Rightarrow \boxed{V_A = 4V_B}$

same heat: $C_A = C_B$

same heat transfer coefficient: $h_A = h_B$

surface area: $A_A = 2A_B$

So: $\tau_A = \frac{\rho_A V_A C_A}{h_A A_A} = \frac{\rho_B \cdot 4V_B \cdot C_B}{h_B \cdot 2A_B} = \frac{2\rho_B V_B C_B}{h_B \cdot A_B} = 2\tau_B$

$\Rightarrow \frac{1}{\tau_A} = \frac{1}{2}\frac{1}{\tau_B}$, thermal drive B has larger bandwidth.

(d) $G(\omega) = \dfrac{0.01}{1 + j\tau\omega}$

corner frequency: $\dfrac{1}{\tau} = 100\ Hz = 100 \times 2\pi\ rad/s$

$$\Rightarrow \tau = \frac{1}{100 \times 2\pi}$$

$$G(\omega) = \frac{0.01}{1 + j\tau\omega} = \frac{0.01}{1 + j\tau\omega} \cdot \frac{1 - j\tau\omega}{1 - j\tau\omega}$$

$$= \frac{0.01}{1 + \tau^2\omega^2} - \frac{\tau\omega}{1 + \tau^2\omega^2} j$$

$$|G(\omega)| = \sqrt{\left(\frac{0.01}{1 + \tau^2\omega^2}\right)^2 + \left(-\frac{\tau\omega}{1 + \tau^2\omega^2}\right)^2}$$

$$= \frac{0.01}{\sqrt{1 + \tau^2\omega^2}}$$

If $|G(\omega)| < 10^{-3}$, NO response can be measured!

$$\Rightarrow \frac{0.01}{\sqrt{1 + \tau^2\omega^2}} < 10^{-3}$$

$$\Rightarrow \frac{10}{\sqrt{1 + \tau^2\omega^2}} < 1$$

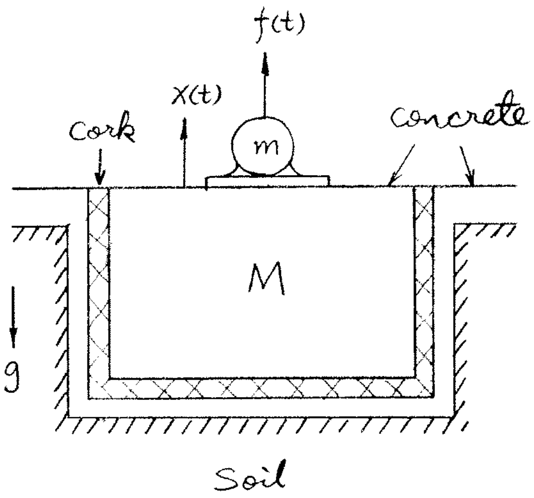
$$\Rightarrow 100 < 1 + \tau^2\omega^2$$

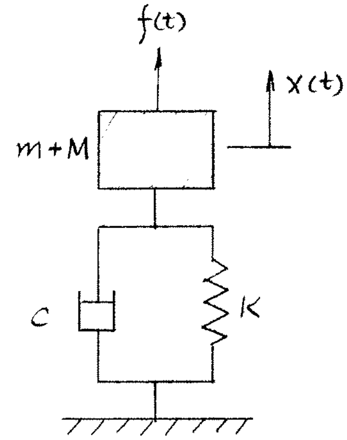
$$\Rightarrow \omega > \frac{\sqrt{99}}{\tau} = 100 \times 2\pi\sqrt{99}\ rad/s$$

$$= 100\sqrt{99}\ Hz \approx 1000\ Hz$$

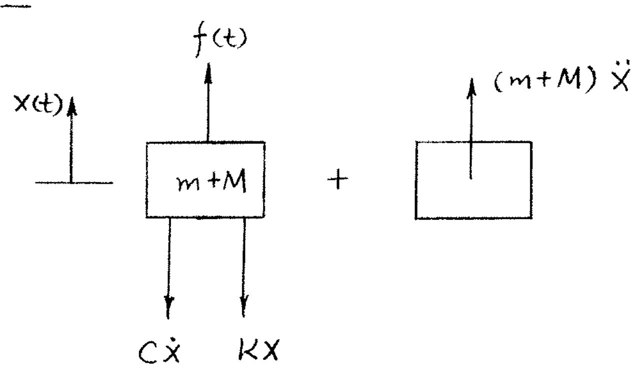
ANS

Problem 4:





(a) F.B.D.



$$f - c\dot{x} - Kx = (m+M)\ddot{x}$$

$$\Rightarrow (m+M)\ddot{x} + c\dot{x} + Kx = f = F_0 \cos \omega t$$

$$\omega_n = \sqrt{\frac{K}{m+M}}$$

$$\zeta = \frac{c}{2(m+M)\omega_n} = \frac{c}{2\sqrt{K(m+M)}}$$

ANS

(b) At equilibrium position, spring force is equal to the weight of foundation mass

$$\delta K = (m+M) g$$

$$\Rightarrow \frac{K}{m+M} = \frac{g}{\delta} = \omega_n^2$$

$$\Rightarrow \omega_n = \sqrt{\frac{g}{\delta}}$$

ANS

(c) $$(m+M)\ddot{x} + c\dot{x} + Kx = F_0 e^{i\omega t}$$

$$x(t) = X(s) e^{i\omega t}$$

$$\Rightarrow (m+M)(i\omega)^2 X(i\omega)\cancel{e^{i\omega t}} + C(i\omega)X(i\omega)\cancel{e^{i\omega t}} + K X(i\omega)\cancel{e^{i\omega t}} = F_0 \cancel{e^{i\omega t}}$$

$$\Rightarrow \left[-(m+M)\omega^2 + c\omega i + K\right] X(i\omega) = F_0(i\omega)$$

$$\Rightarrow G(\omega) = H(i\omega) = \frac{X(i\omega)}{F_0(i\omega)} = \frac{1}{-(m+M)\omega^2 + K + C\omega i}$$

$$\omega_n = \sqrt{\frac{K}{m+M}}, \quad \zeta = \frac{c}{2\sqrt{K(m+M)}}$$

$$\Rightarrow G(\omega) = \frac{1}{K} \frac{1}{1 - \frac{(m+M)}{K}\omega^2 + \frac{c\omega}{K} i}$$

$$= \frac{1}{K} \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2\zeta\omega\sqrt{\frac{m+M}{K}}\, i}$$

$$= \frac{1}{K} \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)i}$$

ANS

(d) $F_T = c\dot{x} + Kx$

$$\Rightarrow F_T(i\omega)\cancel{e^{i\omega t}} = c(i\omega)X(i\omega)\cancel{e^{i\omega t}} + KX(i\omega)\cancel{e^{i\omega t}}$$

$$\Rightarrow F_T(i\omega) = (K + c\omega i)X(i\omega)$$

Transmissibility:

$$G(i\omega) = \frac{F_T(i\omega)}{F_o(i\omega)} = \frac{(K + c\omega i)X(i\omega)}{F_o(i\omega)}$$

$$= \frac{\frac{1}{K}(K + c\omega i)}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)i}$$

$$= \frac{1 + \frac{c\omega}{K}i}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)i}$$

$$= \frac{1 + 2\zeta\left(\frac{\omega}{\omega_n}\right)i}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)i}$$

$$= \frac{1 + 2\zeta\left(\frac{\omega}{\omega_n}\right)i}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + 2\zeta\left(\frac{\omega}{\omega_n}\right)i} \cdot \frac{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] - 2\zeta\left(\frac{\omega}{\omega_n}\right)i}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] - 2\zeta\left(\frac{\omega}{\omega_n}\right)i}$$

$$= \frac{1 + (4\zeta^2 - 1)\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} - \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)^3}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} i$$

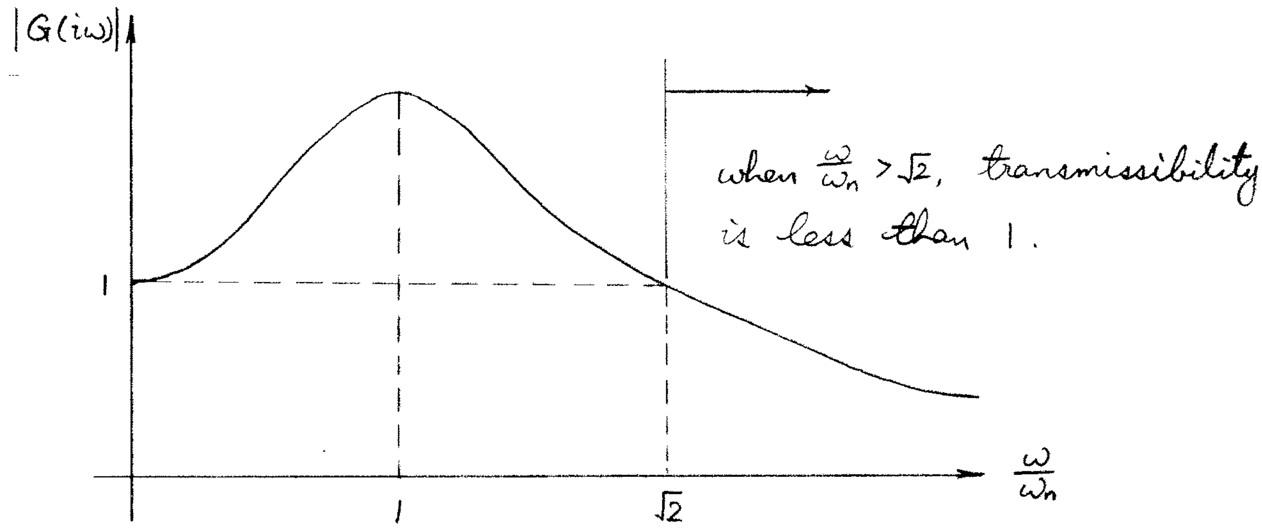
$$|G(i\omega)| = \sqrt{\left(\frac{1+(4\zeta^2-1)(\frac{\omega}{\omega_n})^2}{[1-(\frac{\omega}{\omega_n})^2]^2+4\zeta^2(\frac{\omega}{\omega_n})^2}\right)^2+\left(-\frac{2\zeta(\frac{\omega}{\omega_n})^3}{[1-(\frac{\omega}{\omega_n})^2]^2+4\zeta^2(\frac{\omega}{\omega_n})^2}\right)^2}$$

$\boxed{OR}$:

$$G(i\omega) = \frac{1+2\zeta(\frac{\omega}{\omega_n})i}{1-(\frac{\omega}{\omega_n})^2+2\zeta(\frac{\omega}{\omega_n})i} = \frac{G_1(i\omega)}{G_2(i\omega)}$$

So:

$$|G(i\omega)| = \frac{|G_1(i\omega)|}{|G_2(i\omega)|} = \sqrt{\frac{1+4\zeta^2(\frac{\omega}{\omega_n})^2}{[1-(\frac{\omega}{\omega_n})^2]^2+4\zeta^2(\frac{\omega}{\omega_n})^2}}$$



when $\frac{\omega}{\omega_n} = 0 \Rightarrow |G(i\omega)| = 1$

when $\frac{\omega}{\omega_n} = 1 \Rightarrow |G(i\omega)| = \sqrt{\frac{1+4\zeta^2}{4\zeta^2}}$ (maximum)

@ May 8, 2006 2:07AM