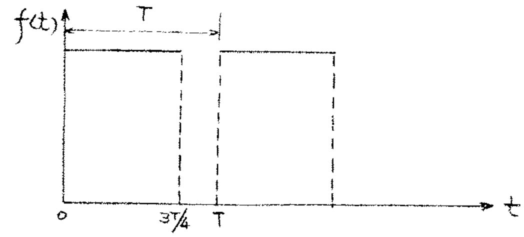
Problem 1

$$f(t) = \begin{cases} 1, & 0 < t < 3T/4 \\ 0, & 3T/4 < t < T \end{cases}$$



(a) Complex Fourier Series. Find C_0 to C_3

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_0^{3T/4} 1 \cdot e^{-jn\omega_0 t}\, dt$$

✓ when $n = 0$,

$$C_0 = \frac{1}{T}\int_0^{3T/4} \underbrace{e^0}_{\to 1}\, dt = \frac{1}{T}\left[t\right]_0^{3T/4} = \frac{1}{T}\left(\frac{3T}{4} - 0\right) = \frac{3}{4} \quad \text{ANS.}$$

✓ when $n \neq 0$,

$$C_n = \frac{1}{T}\int_0^{3T/4} e^{-jn\omega_0 t}\, dt = \frac{1}{T}\,\frac{1}{-jn\omega_0}\int_0^{3T/4} d\, e^{-jn\omega_0 t}$$

$$= \frac{1}{T}\,\frac{j}{n\frac{2\pi}{T}}\left[e^{-jn\frac{2\pi}{T}t}\right]_0^{3T/4} = \frac{j}{2n\pi}\left(e^{-jn\frac{2\pi}{T}\frac{3T}{4}} - 1\right)$$

$$= \frac{1}{2n\pi}\left(e^{-j\frac{3n\pi}{2}} - 1\right)$$

So: $n = 1$

$$C_1 = \frac{j}{2\pi}\left(e^{-j\frac{3\pi}{2}} - 1\right), \qquad \text{Euler Formula: } e^{-j\omega} = \cos\omega - j\sin\omega$$

$$= \frac{j}{2\pi}\left(\underbrace{\cos\frac{3\pi}{2}}_{\to 0} - j\sin\frac{3\pi}{2} - 1\right)$$

$$= \frac{j}{2\pi}\left(j - 1\right)$$

$$= -\frac{1}{2\pi} - \frac{1}{2\pi}j \quad \text{ANS.}$$

$n = 2$,

$$C_2 = \frac{j}{2\cdot 2\pi}\left(e^{-j\frac{3\cdot 2\pi}{2}} - 1\right)$$

$$= \frac{j}{4\pi}\left(\underbrace{\cos 3\pi}_{\to -1} - j\underbrace{\sin 3\pi}_{\to 0} - 1\right)$$

$$= -\frac{1}{2\pi}j \quad \text{ANS.}$$

$n = 3$

$$C_3 = \frac{j}{2\cdot 3\pi}\left(e^{-j\frac{3\cdot 3\pi}{2}} - 1\right)$$

$$= \frac{j}{6\pi}\left(\underbrace{\cos\frac{9\pi}{2}}_{\to 0} - j\sin\frac{9\pi}{2} - 1\right)$$

$$= \frac{j}{6\pi}\left(-j - 1\right)$$

$$= \frac{1}{6\pi} - \frac{1}{6\pi}j \quad \text{ANS.}$$

(b) when $n \neq 0$, C_n and C_{-n} are complex conjugates.

So:

$$C_{-1} = -\frac{1}{2\pi} + \frac{1}{2\pi}j$$

$$C_{-2} = \frac{1}{2\pi}j$$

$$C_{-3} = \frac{1}{6\pi} + \frac{1}{6\pi}j$$

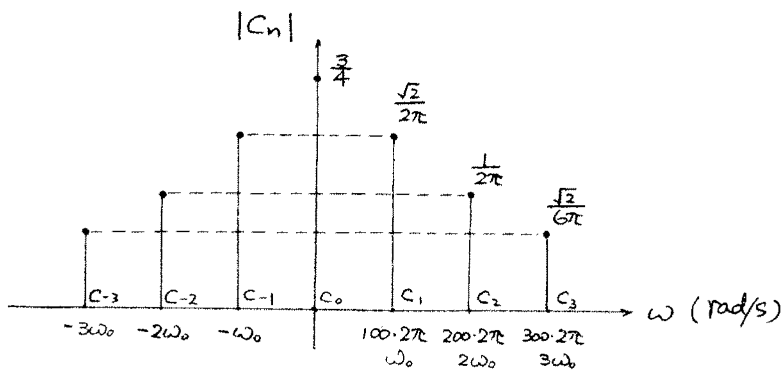
(c) $T = 0.01$ $\qquad \omega_0 = \frac{2\pi}{T} = 100\cdot 2\pi$ rad/s.

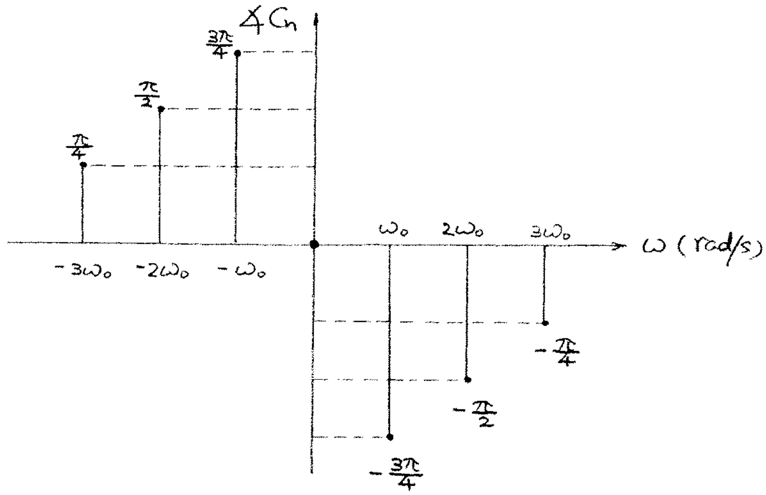
$|C_0| = \frac{3}{4}$, $\qquad \measuredangle C_0 = 0$

$$|C_1| = \sqrt{\left(-\frac{1}{2\pi}\right)^2 + \left(-\frac{1}{2\pi}\right)^2} = \frac{\sqrt{2}}{2\pi}, \qquad \angle C_1 = -\frac{3\pi}{4}$$

$$|C_2| = \sqrt{\left(-\frac{1}{2\pi}\right)^2} = \frac{1}{2\pi}, \qquad \angle C_2 = -\frac{\pi}{2}$$

$$|C_3| = \sqrt{\left(\frac{1}{6\pi}\right)^2 + \left(-\frac{1}{6\pi}\right)^2} = \frac{\sqrt{2}}{6\pi}, \qquad \angle C_3 = -\frac{\pi}{4}$$





Problem 2

(a) $$f(t) = \cos 1.5pt + 2\sin 2.5pt$$

Function is now in Fourier Series, we need to change it to complex Fourier Series.

Recall Euler formula:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \quad \text{—} \ ①$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t \quad \text{—} \ ②$$

$$(① + ②)/2 \Rightarrow \underline{\cos\omega t = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

$$(① - ②)/2j \Rightarrow \sin\omega t = \tfrac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

So:

$$f(t) = \tfrac{1}{2}(e^{j1.5pt} + e^{-j1.5pt}) + \tfrac{2}{2j}(e^{j2.5pt} - e^{-j2.5pt})$$

$$= \tfrac{1}{2}(e^{j1.5pt} + e^{-j1.5pt}) - j(e^{j2.5pt} - e^{-j2.5pt})$$

$$= \tfrac{1}{2}e^{j1.5pt} - je^{j2.5pt} + \tfrac{1}{2}e^{-j1.5pt} + je^{-j2.5pt}$$

✓ Compare with general form:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t} + C_4 e^{j4\omega_0 t} + C_5 e^{j5\omega_0 t} + \cdots$$

$$+ C_{-1} e^{-j\omega_0 t} + C_{-2} e^{-j2\omega_0 t} + C_{-3} e^{-j3\omega_0 t} + C_{-4} e^{-j4\omega_0 t} + C_{-5} e^{-j5\omega_0 t} + \cdots$$

So: $1.5pt = 3\omega_0 t$

$2.5pt = 5\omega_0 t$ $\Rightarrow$ $\underline{\omega_0 = 0.5P}$ ANS.

$$f(t) = \frac{1}{2}e^{j1.5pt} - je^{j2.5pt} + \frac{1}{2}e^{-j1.5pt} + je^{-j2.5pt}$$

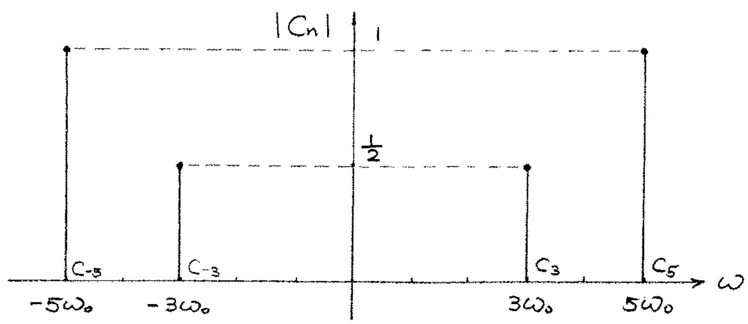
$$= C_3 e^{j3\omega_0 t} + C_5 e^{j5\omega_0 t} + C_{-3} e^{-j3\omega_0 t} + C_{-5} e^{-j5\omega_0 t}$$

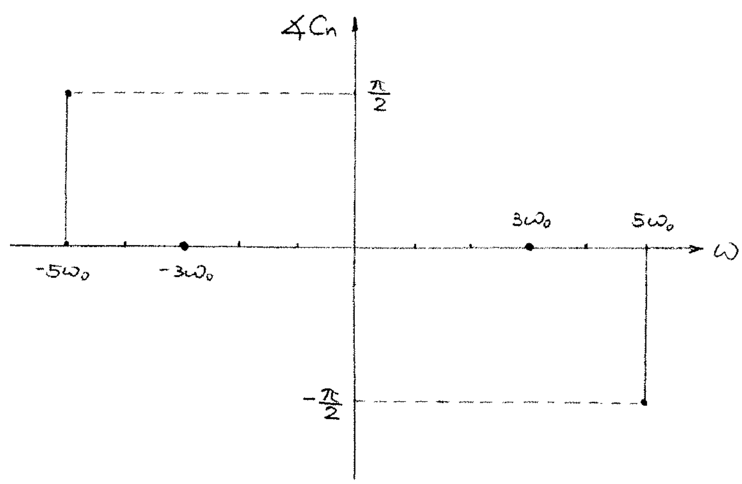
$C_3 = \frac{1}{2}$, $|C_3| = \frac{1}{2}$, $\angle C_3 = 0$

$C_{-3} = \frac{1}{2}$, $|C_{-3}| = \frac{1}{2}$, $\angle C_{-3} = 0$

$C_5 = -j$, $|C_5| = 1$, $\angle C_5 = -\frac{\pi}{2}$

$C_{-5} = j$, $|C_{-5}| = 1$, $\angle C_{-5} = \frac{\pi}{2}$





(b)

$$G(\omega) = \frac{j\omega}{m(\omega_n^2 - \omega^2)}, \quad \omega_n = 2p$$

Assume output has a form:

$$x(t) = d_3 e^{j1.5pt} + d_5 e^{j2.5pt} + d_{-3} e^{-j1.5pt} + d_{-5} e^{-j2.5pt}$$

$$d_3 = G(1.5p) \cdot C_3 = \frac{j1.5p}{m[4p^2 - (1.5p)^2]} \cdot \frac{1}{2} = \frac{0.429}{mp} j$$

$$d_5 = G(2.5p)\, C_5 = \frac{j2.5p}{m[4p^2 - (2.5p)^2]} \cdot (-j) = -\frac{1.111}{mp}$$

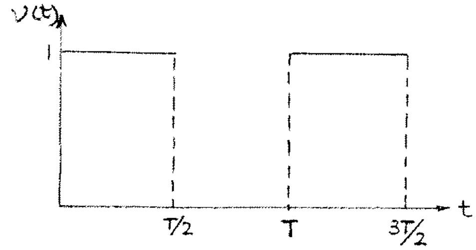
So:

$$x(t) = \frac{0.429}{mp} j e^{j1.5pt} - \frac{1.111}{mp} e^{j2.5pt} - \frac{0.429}{mp} j e^{-j1.5pt} - \frac{1.111}{mp} e^{-j2.5pt}$$

$$= \frac{1}{mp}(-0.858 \sin 1.5pt - 2.222 \cos 2.5pt)$$

ANS

Problem 3



(a)

$$v(t) = \begin{cases} 1, & 0 < t < T/2 \\ 0, & \text{Otherwise} \end{cases}$$

Complex Fourier Series

$$v(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

$$C_n = \frac{1}{T}\int_0^T v(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T}\int_0^{T/2} 1 \cdot e^{-jn\omega_0 t} dt$$

✓ when $n = 0$

$$C_0 = \frac{1}{T}\int_0^{T/2} \underbrace{e^{0}}_{\to 1} dt = \frac{1}{T}\left[t\right]_0^{T/2} = \frac{1}{T}\left(\frac{T}{2} - 0\right) = \underline{\frac{1}{2}} \text{ ANS.}$$

✓ when $n \neq 0$

$$C_n = \frac{1}{T}\int_0^{T/2} e^{-jn\omega_0 t} dt = \frac{1}{T}\,\frac{1}{-jn\omega_0}\left[e^{-jn\omega_0 t}\right]_0^{T/2}$$

$$= \frac{1}{\cancel{T}}\,\frac{j}{n\frac{2\pi}{\cancel{T}}}\left[e^{-jn\frac{\cancel{2}\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}} - 1\right] = \frac{j}{2n\pi}\left(e^{-jn\pi} - 1\right)$$

$n = 1, \quad \omega = \omega_0$

$$C_1 = \frac{j}{2\pi}\left(e^{-j\pi} - 1\right) = \frac{j}{2\pi}\left(\underbrace{\cos\pi}_{\to -1} - j\underbrace{\sin\pi}_{\to 0} - 1\right)$$

$$= \frac{j}{2\pi}(-2) = \underline{-\frac{1}{\pi} j} \text{ ANS.}$$

$n = 2, \quad \omega = 2\omega_0$

$$C_2 = \frac{j}{2 \cdot 2\pi}\left(e^{-j2\pi} - 1\right) = \frac{j}{4\pi}\left(\underbrace{\cos 2\pi}_{\to 1} - j\underbrace{\sin 2\pi}_{\to 0} - 1\right)$$

$$= \frac{j}{4\pi} \times 0 = \underline{0} \quad \text{Ans.}$$

$$C_n = \frac{j}{2n\pi}(e^{-jn\pi} - 1) = \frac{j}{2n\pi}(\cos n\pi - j\underbrace{\sin n\pi}_{\to 0} - 1)$$

$$= \frac{j}{2n\pi}(\cos n\pi - 1) = \begin{cases} -\frac{j}{n\pi}, & n = \text{odd number} \\ 0, & n = \text{even number} \end{cases}$$

$n = 3, \quad \omega = 3\omega_0$

$$\underline{C_3 = -\frac{j}{3\pi}} \quad \text{Ans.}$$

✓ Complex Conjugates:

$$C_{-1} = \frac{1}{\pi} j, \quad C_{-2} = 0, \quad C_{-3} = \frac{1}{3\pi} j \quad \text{Ans.}$$

(b)

$$|C_n| = \begin{cases} \frac{1}{2}, & n = 0 \\ \left|\frac{1}{n\pi}\right|, & n = \text{odd number} \\ 0, & n = \text{even number} \end{cases}$$

$$\measuredangle C_n = \begin{cases} 0, & n = 0 \\ -\frac{\pi}{2}, & n = \text{positive odd number} \\ \frac{\pi}{2}, & n = \text{negative odd number} \\ 0, & n = \text{even number} \end{cases}$$

So:

$|C_0| = \frac{1}{2}$, $\measuredangle C_0 = 0$

$|C_1| = \frac{1}{\pi}$, $\measuredangle C_1 = -\frac{\pi}{2}$

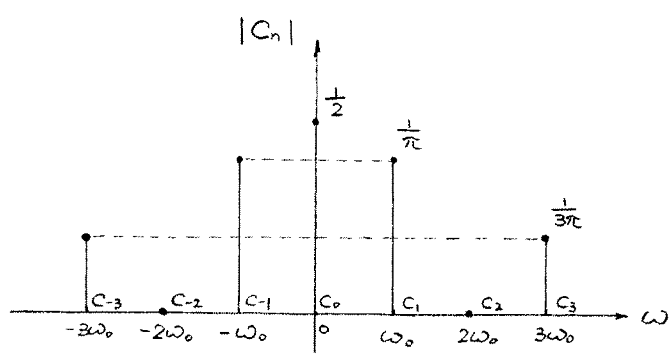
$|C_2| = 0$, $\measuredangle C_2 = 0$

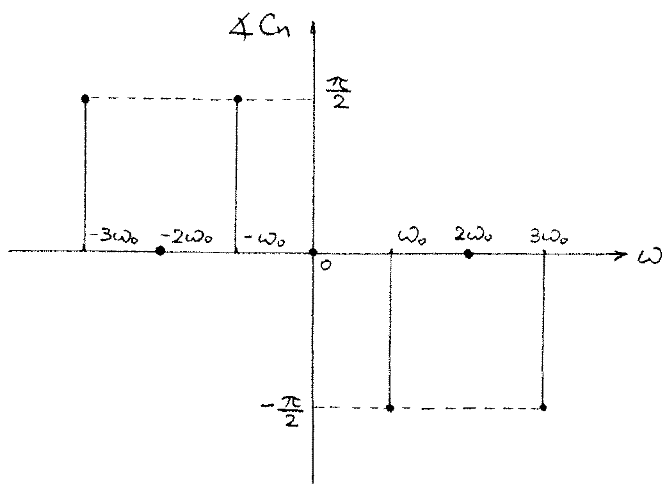
$|C_3| = \frac{1}{3\pi}$, $\measuredangle C_3 = -\frac{\pi}{2}$

$|C_{-1}| = \frac{1}{\pi}$, $\measuredangle C_{-1} = \frac{\pi}{2}$

$|C_{-2}| = 0$, $\measuredangle C_{-2} = 0$

$|C_{-3}| = \frac{1}{3\pi}$, $\measuredangle C_{-3} = \frac{\pi}{2}$





(C) $T = 1\ ms \Rightarrow \omega_o = \frac{2\pi}{T} = 1000 \times 2\pi$ rad/s.

$\frac{1}{RC} = 500\ Hz = 500 \times 2\pi$ rad/s

$\omega_o > \frac{1}{RC}$.

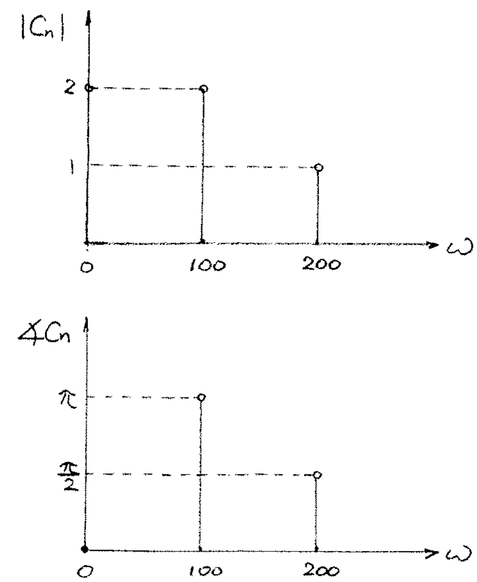
So all input frequencies are located in flat zone of FRF, amplitude of each input will be magnified about same value, So there will be NO distorsion and output will look like the same form as input.

Problem 4.

(a) $$G(\omega) = \frac{j\omega}{1+j\omega\tau}, \quad \tau = 1 \text{ second}$$

$$\Rightarrow \quad G(\omega) = \frac{j\omega}{1+j\omega}$$

✓ From spectum on right



$|C_0| = 2, \quad \measuredangle C_0 = 0$

$|C_1| = 2, \quad \measuredangle C_1 = \pi$

$|C_2| = 1, \quad \measuredangle C_2 = \frac{\pi}{2}$

So:

$$C_0 = 2e^{j0} = 2$$

$$C_1 = 2e^{j\pi} = 2(\underbrace{\cos\pi}_{\to -1} + j\,\underbrace{\sin\pi}_{\to 0}) = -2, \qquad C_{-1} = -2$$

$$C_2 = 1 \cdot e^{j\frac{\pi}{2}} = \underbrace{\cos\frac{\pi}{2}}_{\to 0} + j\sin\frac{\pi}{2} = j, \qquad C_{-2} = -j$$

Assume output with form:

$$x(t) = d_0 + d_1 e^{j100t} + d_2 e^{j200t} + d_{-1} e^{-j100t} + d_{-2} e^{-j200t}$$

$$d_0 = G(0)\,C_0 = 0$$

$$d_1 = G(100)\,C_1 = \frac{j100}{1+j100} \cdot (-2) \approx -2$$

$$d_2 = G(200)\,C_2 = \frac{j200}{1+j200}\, j \approx j$$

$$d_{-1} = G(-100)\,C_{-1} = \frac{j(-100)}{1+j(-100)}\cdot(-2) \approx -2$$

$$d_{-2} = G(-200)\,C_{-2} = \frac{j(-200)}{1+j(-200)}(-j) \approx -j$$

So:

$$x(t) = -2e^{j100t} + je^{j200t} - 2e^{-j100t} - je^{-j200t}$$

$$= \underline{-4\cos 100t - 2\sin 200t} \quad \text{ANS.}$$

(b) Output will have same form as input when DC term is filtered out.

Problem 5:

(a)
$$I\ddot{\theta}(t) + 2cl^2\dot{\theta}(t) + 2kl^2\theta(t) = aF(t)$$

$$\omega_n = \sqrt{\frac{2kl^2}{I}}, \qquad \zeta = \frac{2cl^2}{2I\omega_n} = \frac{cl^2}{I\omega_n}$$ ANS.

(b) Let $\theta(t) = \Theta(s)e^{st}$, $F(t) = F(s)e^{st}$

$$\Rightarrow Is^2\Theta(s)\cancel{e^{st}} + 2cl^2 s\,\Theta(s)\cancel{e^{st}} + 2kl^2\Theta(s)\cancel{e^{st}} = aF(s)\cancel{e^{st}}$$

$$\Rightarrow \tilde{H}(s) = \frac{\Theta(s)}{F(s)} = \frac{a}{Is^2 + 2cl^2 s + 2kl}$$

$$\alpha(t) = A(s)\cancel{e^{st}} = \ddot{\theta}(t) = s^2\Theta(s)\cancel{e^{st}}$$

$$\Rightarrow H(s) = \frac{A(s)}{F(s)} = \frac{s^2\Theta(s)}{F(s)} = s^2\hat{H}(s) = \frac{s^2 a}{Is^2 + 2cl^2 s + 2kl}$$

So: FRF

$$G(\omega) = \frac{-a\omega^2}{(2kl - I\omega^2) + 2cl^2\omega j}, \qquad c = \frac{\zeta I\omega_n}{l^2}$$ ANS.

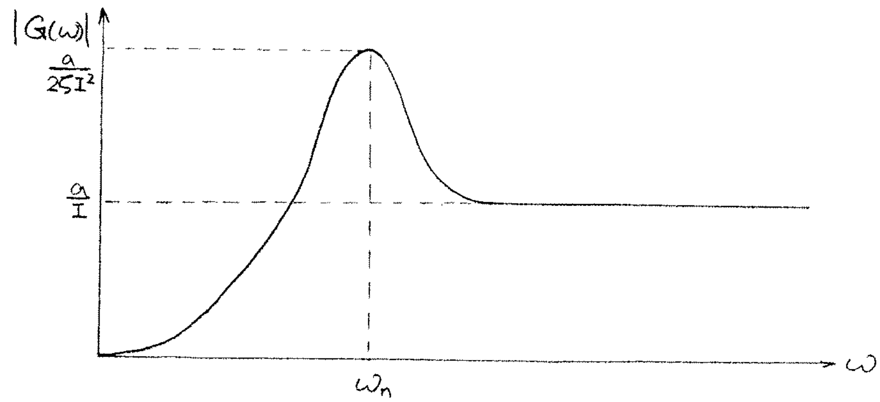
$$|G(\omega)| = \frac{a\omega^2}{\sqrt{(2kl^2 - I\omega^2)^2 + 4c^2l^4\omega^2}} = \frac{a\omega^2}{\sqrt{I^2(\omega_n^2-\omega^2)^2 + 4\zeta^2 I^2\omega_n^2\omega^2}}$$

$$= \frac{a\omega^2}{I\omega_n^2\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2 I^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{a}{I}\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2 I^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

when $\omega \ll \omega_n$, $|G(\omega)| \approx \frac{a}{I}\left(\frac{\omega}{\omega_n}\right)^2 = \frac{a}{\cancel{I}}\frac{\cancel{I}}{2kl^2}\omega^2 = \frac{a}{2kl^2}\omega^2$

when $\omega = \omega_n$, $|G(\omega)| = \frac{a}{I}\frac{1}{2\zeta I} = \frac{a}{2\zeta I^2}$

when $\omega \gg \omega_n$, $|G(\omega)| \approx \frac{a}{I} \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2} = \frac{a}{I}$



(c) Input:

$$F(t) = \begin{cases} 1, & 0 < t < \varepsilon \\ 0, & \varepsilon < t < T \end{cases} \qquad F(t+T) = F(t)$$

Fourier Series.

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = \frac{1}{T}\int_0^T F(t)\,dt = \frac{1}{T}\int_0^{\varepsilon} 1\,dt = \underline{\frac{\varepsilon}{T}} \quad \text{ANS.}$$

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_0^{\varepsilon} 1\cdot\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\frac{1}{n\omega_0}\left[\sin n\omega_0 t\right]_0^{\varepsilon} = \underline{\frac{1}{n\pi}\sin n\omega_0 \varepsilon} \quad \text{ANS.}$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_0^{\varepsilon} 1\cdot\sin n\omega_0 t\,dt$$

$$= \frac{2}{T}\frac{-1}{n\omega_0}\left[\cos n\omega_0 t\right]_0^{\varepsilon} = \underline{\frac{1}{n\pi}(1-\cos n\omega_0 \varepsilon)} \quad \text{ANS.}$$

So:

$$F(t) = \frac{\varepsilon}{T} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin n\omega_0\varepsilon \cos n\omega_0 t + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-\cos n\omega_0\varepsilon) \sin n\omega_0 t$$

(d) $\omega_n = 250\,Hz$, $\zeta = 0.01$, $a/I = 1$, $T = 2.5\,ms$, $\varepsilon = 1$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.0025} = 400 \times 2\pi \text{ rad/s}$$

$$\omega_n = 250 \times 2\pi \text{ rad/s}$$

$\omega_0 > \omega_n$, So all input frequencies are above ω_n.

✓ when $\omega \gg \omega_n$

$$G(\omega) = \frac{-a\omega^2}{(2KL - I\omega^2) + 2CL^2\omega j} = \frac{-a\omega^2}{I(\omega_n^2 - \omega^2) + 4\zeta I\omega_n\omega j}$$

$$= \frac{-a}{I} \cdot \frac{\omega^2}{\omega_n^2\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + 4\zeta\left(\frac{\omega}{\omega_n}\right)j\right\}}$$

$$= -\frac{a}{I} \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 4\zeta\left(\frac{\omega}{\omega_n}\right)j}$$

$$\approx -\frac{a}{I} \frac{\cancel{\left(\frac{\omega}{\omega_n}\right)^2}}{-\cancel{\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{a}{I}$$

$$\angle G(\omega) = 0$$

So when all inputs go through $G(\omega)$, there will be no phase change.

Assume output with form:

$$\alpha(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\omega_0 t + \sum_{n=1}^{\infty} B_n \sin n\omega_0 t$$

$$A_0 = G(0)\, a_0 = 0$$

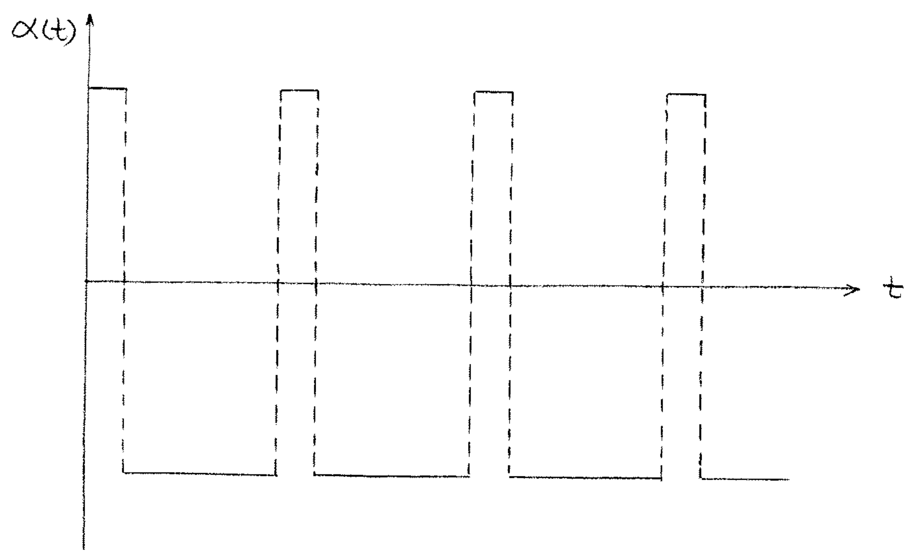
$$A_n = G(n\omega_0)\, a_n \approx \frac{a}{I} a_n = a_n$$

$$B_n = G(n\omega_0)\, b_n \approx \frac{a}{I} b_n = b_n$$

So:

$$\alpha(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin n\omega_0 \varepsilon \cos n\omega_0 t + \sum_{n=1}^{\infty} \frac{1}{n\pi} (1 - \cos n\omega_0 \varepsilon) \sin n\omega_0 t$$



Output will have same form as input when DC term is filtered out.