ME 374, System Dynamics Analysis and Design
Homework 9: Solution (June 9, 2008)
by Jason Frye

Problem 1

(a) The frequency response function $G(\omega)$ and the impulse response function $h(t)$ are Fourier transform pairs. Therefore,

$$G(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

It is reasonable to assume that $h(t)$ will only be considered for $t > 0$, or

$$h(t) = \begin{cases} e^{-t/\tau}, & t > 0 \\ 0, & t < 0 \end{cases}.$$

Therefore,

$$\begin{aligned} G(\omega) &= \int_0^{\infty} e^{-t/\tau}e^{-j\omega t}dt \\ &= \int_0^{\infty} e^{-(j\omega+t/\tau)t}dt \\ &= -\frac{1}{j\omega+1/\tau}e^{-(j\omega+t/\tau)t}\Big|_0^{\infty} \\ &= -\frac{1}{j\omega+1/\tau}\left(e^{-(j\omega+t/\tau)(\infty)} - e^{-(j\omega+t/\tau)(0)}\right) \\ &= -\frac{1}{j\omega+1/\tau}(0-1) \\ G(\omega) &= \underline{\underline{\frac{1}{j\omega+1/\tau}}}. \quad (Ans.) \end{aligned}$$

(b) Now we are given the FRF from $y(t)$ to $x(t)$

$$G(\omega) = \frac{1}{1+10\omega j} \tag{2}$$

whose magnitude is given by

$$\underline{\underline{|G(\omega)| = \frac{1}{\sqrt{1+100\omega^2}} = \frac{0.1}{\sqrt{(0.1)^2+\omega^2}}}}. \quad (Ans.)$$

For $\omega \ll 0.1$ (such as $\omega \approx 0$)

$$|G(\omega)| = 1 \quad \text{or} \quad 20\log_{10}\{G(\omega)\} = 0 \text{ dB}.$$

The magnitude behaves like a low-pass filter with cutoff frequency 0.1 rad/s and rolls off at 20 dB/decade, as shown in Figure 1.

(c) The frequency response function given by (2) corresponds to a first-order system since there is only one pole in the denominator. As mentioned, this forms a low-pass filter with cutoff frequency 0.1 rad/s. Therefore, the bandwidth is 0.1 rad/s.

(d) The magnitude of the output spectrum of $x(t)$ (i.e., the magnitude of $X(\omega)$) is shown in Figure 2(a). The frequency response function from the input $y(t)$ to the output $x(t)$ is $G(\omega) = \frac{X(\omega)}{Y(\omega)}$. Therefore, the input can be determined from

$$Y(\omega) = \frac{X(\omega)}{G(\omega)} \quad \text{and} \quad |Y(\omega)| = \frac{|X(\omega)|}{|G(\omega)|} = \frac{1}{|G(\omega)|} \cdot |X(\omega)|.$$

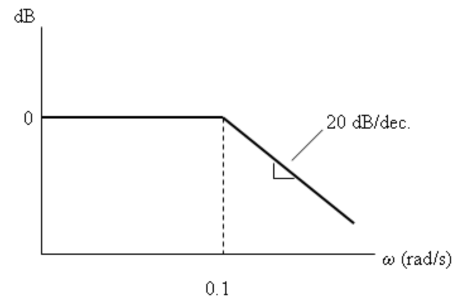


Figure 1: Magnitude of the frequency response function $G(\omega)$.

The magnitude of $1/|G(\omega)|$ is shown in Figure 2(b). Note that the magnitude of $G(\omega)$ and the magnitude of the output spectrum of $x(t)$ are both constant ($= 0$ dB) up to their cutoff frequency. However, the cutoff frequency for the output spectrum of $x(t)$ is much higher than that for $G(\omega)$. Therefore, the magnitude of the input spectrum of $y(t)$ will initially follow the magnitude of $1/|G(\omega)|$ up to $\omega = 100$ rad/s. At that point, the magnitude of the input spectrum of $y(t)$ is the combination of the magnitude of $1/|G(\omega)|$ (+20 dB/decade) and the magnitude of the output spectrum of $x(t)$ (-40 dB/decade), or a net roll off of -20 dB/decade. The magnitude of the input spectrum of $y(t)$ is shown in Figure 3.

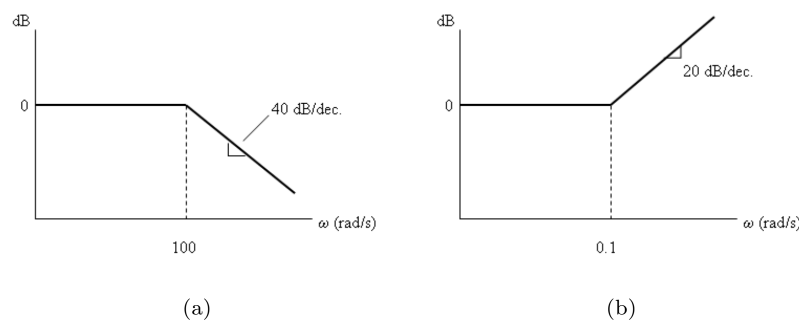


Figure 2: (a) Magnitude of the output spectrum of $x(t)$; (b) Magnitude of $1/|G(\omega)|$.

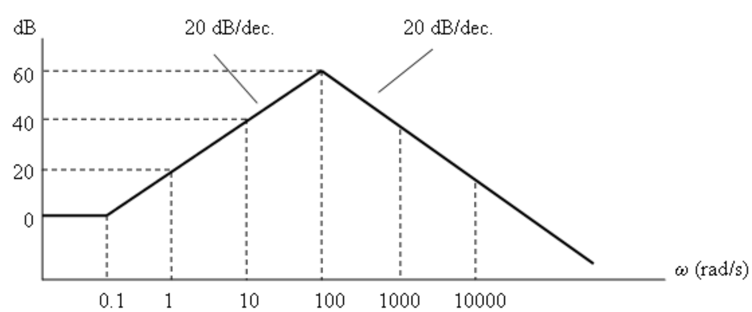


Figure 3: Magnitude of the input spectrum of $y(t)$.

Problem 2

(a) For the given force history

$$f(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

generated by the hammer, where T is the duration of the hammer impact, the Fourier transform $F(\omega)$ is

$$\begin{aligned} F(\omega) = \mathcal{F}\{f(t)\} &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ &= \int_0^T e^{-j\omega t} dt \\ &= -\frac{1}{j\omega} e^{-j\omega t}\Big|_0^T \\ F(\omega) &= \underline{\underline{\frac{j}{\omega}\left(e^{-j\omega T} - 1\right)}}. \quad (Ans.) \end{aligned}$$

(b) The amplitude of $F(\omega)$ is determined by first rewriting $F(\omega)$ as

$$\begin{aligned} F(\omega) = \frac{j}{\omega}\left(e^{-j\omega T} - 1\right) &= \frac{j}{\omega}\left(\cos(\omega T) - j\sin(\omega T) - 1\right) \\ &= \frac{1}{\omega}\left(\sin(\omega T) + j\left(\cos(\omega T) - 1\right)\right) \end{aligned}$$

Then the amplitude is

$$\begin{aligned} |F(\omega)| &= \frac{1}{\omega}\sqrt{\sin^2(\omega T) + \left(\cos(\omega T) - 1\right)^2} \\ &= \frac{1}{\omega}\sqrt{\sin^2(\omega T) + \cos^2(\omega T) - 2\cos(\omega T) + 1} \\ |F(\omega)| &= \underline{\underline{\frac{1}{\omega}\sqrt{2 - 2\cos(\omega T)}}} \quad (Ans.) \end{aligned}$$

which is illustrated in Figure 4. Note that when $\omega = \frac{2\pi}{T}, \frac{4\pi}{T}, \ldots, \frac{2n\pi}{T}$ $(n = 1,\ 2,\ 3, \ldots)$,

$$|F(\omega)| = \frac{1}{(2n\pi/T)}\sqrt{2 - 2\cos\left(\frac{2n\pi}{T}T\right)} = 0.$$

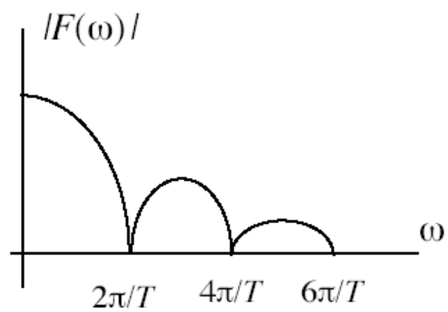


Figure 4: Amplitude of $F(\omega)$.

(c) Note that the 400-Hz resonance of the disk corresponds to $\omega = 800\pi$ rad/s. If $T = 2.5$ ms $= 0.0025$ s, then

$$\frac{2\pi}{T} = \frac{2\pi\ \text{(rad)}}{0.0025\ \text{(s)}} = 800\pi\ \text{rad/s} = 400\ \text{Hz}.$$

From 2(b), $|F(2\pi/T)| = 0$. If the output of the system is $x(t)$, then output spectrum of $x(t)$ is

$$X(\omega) = G(\omega)F(\omega) \quad \text{and} \quad |X(\omega)| = |G(\omega)| \cdot |F(\omega)|.$$

Since $\underline{\underline{|F(400\text{ Hz})| = 0,\ |X(400\text{ Hz})| = 0}}$ as well. Therefore, $\underline{\underline{\text{Henry would } \textbf{not} \text{ excite the 400-Hz resonance of the disk}}}$.

At 500 Hz, $\underline{\underline{|F(500\text{ Hz})| \neq 0, \text{ and } |G(500\text{ Hz})| = 1.5}}$. Therefore, $\underline{\underline{\text{Henry would excite the 500-Hz resonance of the disk}}}$.

(d) Now $T = 1$ ms $= 0.001$ s.

At $\underline{\omega = 400 \text{ Hz} = 800\pi \text{ rad/s}}$:

$$\begin{aligned} |F(400\text{ Hz})| = |F(800\pi)| &= \frac{1}{800\pi}\sqrt{2 - 2\cos(800\pi \cdot 0.001)} \\ &= 7.57 \cdot 10^{-4}. \end{aligned}$$

Then

$$|X(400\text{ Hz})| = |G(400\text{ Hz})| \cdot |F(400\text{ Hz})| = (1) \cdot (7.57 \cdot 10^{-4})$$

or

$$\underline{\underline{|X(400\text{ Hz})| = 7.57 \cdot 10^{-4}}}. \quad (Ans.)$$

At $\underline{\omega = 500 \text{ Hz} = 1000\pi \text{ rad/s}}$:

$$\begin{aligned} |F(500\text{ Hz})| = |F(1000\pi)| &= \frac{1}{1000\pi}\sqrt{2 - 2\cos(1000\pi \cdot 0.001)} \\ &= \frac{1}{500\pi}. \end{aligned}$$

Then

$$|X(500\text{ Hz})| = |G(500\text{ Hz})| \cdot |F(500\text{ Hz})| = (1.5) \cdot \left(\frac{1}{500\pi}\right)$$

or

$$\underline{\underline{|X(500\text{ Hz})| = \frac{1.5}{500\pi}}}. \quad (Ans.)$$

Problem 3

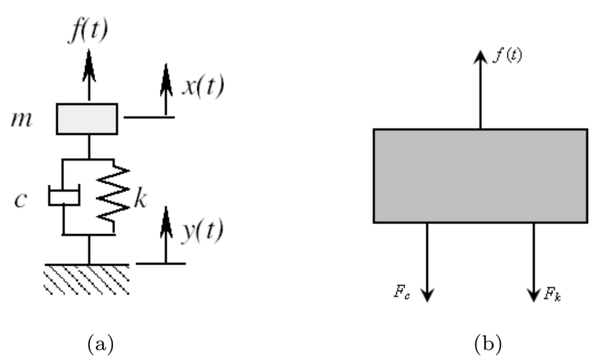


Figure 5: (a) Model for vibration control; (b) Free-body diagram.

(a) For the system shown in Figure 5(a), the equation of motion can be determined using the free-body diagram in Figure 5(b). Because $x(t)$ and $y(t)$ are taken as absolute displacements, the forces F_c and F_k are

$$F_c = c(\dot{x} - \dot{y}), \quad F_k = k(x - y).$$

Then, summing forces gives

$$\begin{aligned}\sum F &= f(t) - F_c - F_k = m\ddot{x} \\ &= f(t) - c(\dot{x} - \dot{y}) - k(x - y) = m\ddot{x}\end{aligned}$$

or

$$\underline{\underline{m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky + f(t)}}. \quad (Ans.)$$

(b) When the system is passive ($f(t) = 0$), the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky.$$

The frequency response function $G(\omega)$ from $y(t)$ to $x(t)$ is

$$\underline{\underline{G(\omega) = \frac{X(\omega)}{Y(\omega)} = \frac{k + jc\omega}{k - m\omega^2 + jc\omega}}}. \quad (Ans.)$$

This can be written using the natural frequency ω_n and damping coefficient ζ as

$$G(\omega) = \frac{\omega_n^2 + j2\zeta\omega_n\omega}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega} = \frac{1 + j2\zeta(\frac{\omega}{\omega_n})}{1 - (\frac{\omega}{\omega_n})^2 + j2\zeta(\frac{\omega}{\omega_n})},$$

where

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n}.$$

Then, the amplitude of $G(\omega)$ is

$$|G(\omega)| = \sqrt{\frac{1 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}.$$

The amplitude near resonance can be determined by letting $\omega = \omega_n$, which gives

$$\underline{\underline{|G(\omega_n)| = \sqrt{\frac{1 + 4\zeta^2}{4\zeta^2}}}}. \quad (Ans.)$$

The rate at which $|G(\omega)|$ rolls off for $\omega \gg \omega_n$ is

$$\underline{\underline{|G(\omega)| \approx \sqrt{\frac{4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^4}} \approx \frac{1}{\omega}}}. \quad (Ans.)$$

The amplitude of $G(\omega)$ is shown in Figure 6.

(c) Now, with spring-force cancellation applied ($f(t) = -ky$), the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = c\dot{y}.$$

The frequency response function $G_{sp}(\omega)$ is then

$$\underline{\underline{G_{sp}(\omega) = \frac{jc\omega}{k - m\omega^2 + jc\omega} = \frac{j2\zeta\omega_n\omega}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega}}}, \quad (Ans.)$$

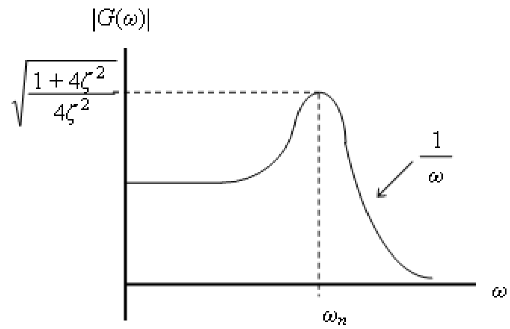


Figure 6: Amplitude of $G(\omega)$.

where again

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n}.$$

The amplitude of $G_{sp}(\omega)$ is

$$|G_{sp}(\omega)| = \frac{2\zeta\omega_n\omega}{\sqrt{\left(\omega_n^2-\omega^2\right)^2+4\zeta^2\omega_n^2\omega^2}} = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}.$$

The amplitude near resonance can be determined by letting $\omega = \omega_n$, which gives

$$\underline{\underline{|G_{sp}(\omega_n)| = \frac{2\zeta}{\sqrt{4\zeta^2}} = 1.}} \quad (Ans.)$$

The rate at which $|G_{sp}(\omega)|$ rolls off for $\omega \gg \omega_n$ is

$$\underline{\underline{|G_{sp}(\omega)| \approx \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\sqrt{\left(\frac{\omega}{\omega_n}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \approx \frac{1}{\omega}.}} \quad (Ans.)$$

The amplitude of $G_{sp}(\omega)$ is shown in Figure 7.

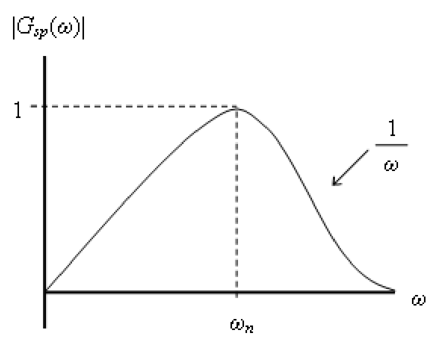


Figure 7: Amplitude of $G_{sp}(\omega)$.

(d) Now, with damping-force cancellation applied ($f(t) = -c\dot{y}$), the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = ky.$$

The frequency response function $G_{damp}(\omega)$ is then

$$\underline{\underline{G_{damp}(\omega) = \frac{k}{k - m\omega^2 + jc\omega} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega}}}, \quad (Ans.)$$

where again

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n}.$$

The amplitude of $G_{damp}(\omega)$ is

$$|G_{damp}(\omega)| = \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega_n^2\omega^2}} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}.$$

The amplitude near resonance can be determined by letting $\omega = \omega_n$, which gives

$$\underline{\underline{|G_{damp}(\omega_n)| = \frac{1}{\sqrt{4\zeta^2}} = \frac{1}{2\zeta}}}. \quad (Ans.)$$

The rate at which $|G_{damp}(\omega)|$ rolls off for $\omega \gg \omega_n$ is

$$\underline{\underline{|G_{damp}(\omega)| \approx \frac{1}{\sqrt{\left(\frac{\omega}{\omega_n}\right)^4}} \approx \frac{1}{\omega^2}}}. \quad (Ans.)$$

The amplitude of $G_{damp}(\omega)$ is shown in Figure 8.

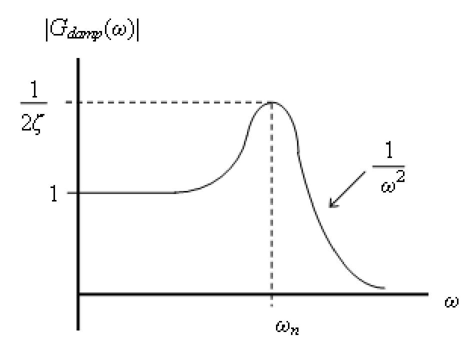


Figure 8: Amplitude of $G_{damp}(\omega)$.

(e) During an earthquake, the goal is to minimize the displacement of the building relative to the ground when excited at resonance. When applying spring-force cancellation, $|G_{sp}(\omega_n)| = 1$. Therefore, $|X(\omega_n)| = |Y(\omega_n)|$ (i.e., the building moves with the ground). When applying damping-force cancellation, $|G_{damp}(\omega_n)| = \frac{1}{2\zeta} > 1$. Therefore, you would use spring-force cancellation.

(f) For an isolation table, the goal is to minimize the absolute displacement of the table. When applying spring-force cancellation, $|G_{sp}(\omega)|$ rolls off as $\frac{1}{\omega}$ for $\omega \gg \omega_n$. When applying damping-force cancellation, $|G_{damp}(\omega)|$ rolls off as $\frac{1}{\omega^2}$ for $\omega \gg \omega_n$. By ensuring that $\omega \gg \omega_n$, you would use damping-force cancellation.

(g) The Fourier transform of

$$f(t) = \begin{cases} \sin(\omega_0 t), & 0 < t < \frac{2\pi}{\omega_0} \\ 0, & \text{otherwise} \end{cases}$$

is

$$\begin{aligned}
F(\omega) = \mathcal{F}\{f(t)\} &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\
&= \int_0^{\frac{2\pi}{\omega_0}} \sin(\omega_0 t) e^{-j\omega t} dt \\
&= \int_0^{\frac{2\pi}{\omega_0}} \frac{1}{2j} \left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-j\omega t} dt \\
&= \frac{1}{2j} \int_0^{\frac{2\pi}{\omega_0}} \left(e^{j(\omega_0-\omega)t} - e^{-j(\omega_0+\omega)t}\right) dt \\
&= \frac{1}{2j} \left[\frac{1}{j(\omega_0-\omega)} e^{j(\omega_0-\omega)t} + \frac{1}{j(\omega_0+\omega)} e^{-j(\omega_0+\omega)t}\right]\Bigg|_0^{\frac{2\pi}{\omega_0}} \\
&= -\frac{1}{2(\omega_0-\omega)} \left(e^{j(\omega_0-\omega)\frac{2\pi}{\omega_0}} - 1\right) - \frac{1}{2(\omega_0+\omega)} \left(e^{-j(\omega_0+\omega)\frac{2\pi}{\omega_0}} - 1\right) \\
&= -\frac{1}{2(\omega_0-\omega)} \left(e^{j2\pi} e^{-j2\pi\frac{\omega}{\omega_0}} - 1\right) - \frac{1}{2(\omega_0+\omega)} \left(e^{-j2\pi} e^{-j2\pi\frac{\omega}{\omega_0}} - 1\right) \\
&= \frac{1}{2(\omega_0-\omega)} \left(1 - e^{-j2\pi\frac{\omega}{\omega_0}}\right) + \frac{1}{2(\omega_0+\omega)} \left(1 - e^{-j2\pi\frac{\omega}{\omega_0}}\right)
\end{aligned}$$

or

$$\underline{\underline{F(\omega) = \frac{\omega_0}{\omega_0^2 - \omega^2} \left(1 - e^{-j2\pi\frac{\omega}{\omega_0}}\right).}} \quad (Ans.)$$

The magnitude of $F(\omega)$ can be determined by first rewriting as

$$\begin{aligned}
F(\omega) &= \frac{\omega_0}{\omega_0^2 - \omega^2} \left[1 - \left(\cos\left(2\pi\frac{\omega}{\omega_0}\right) - j\sin\left(2\pi\frac{\omega}{\omega_0}\right)\right)\right] \\
&= \frac{\omega_0}{\omega_0^2 - \omega^2} \left[1 - \cos\left(2\pi\frac{\omega}{\omega_0}\right) + j\sin\left(2\pi\frac{\omega}{\omega_0}\right)\right].
\end{aligned}$$

Then,

$$|F(\omega)| = \frac{\omega_0}{|\omega_0^2 - \omega^2|} \sqrt{\left(1 - \cos\left(2\pi\frac{\omega}{\omega_0}\right)\right)^2 + \sin^2\left(2\pi\frac{\omega}{\omega_0}\right)}$$

or

$$\underline{\underline{|F(\omega)| = \frac{\omega_0}{|\omega_0^2 - \omega^2|} \sqrt{2 - 2\cos\left(2\pi\frac{\omega}{\omega_0}\right)}.}} \quad (Ans.)$$

Note that $|F(\omega)| = 0$ when $\omega = 2\omega_0, 3\omega_0, \ldots$, as illustrated in Figure 9. Also, in the case when $\omega \to \omega_0$,

$$\lim_{\omega \to \omega_0} |F(\omega)| = \frac{\pi}{\omega_0}.$$

When $\omega \approx \omega_0$ (and if $\omega_0 \gg \omega_n$), the spectrum of $f(t)$ is in the rolloff portion of $|G(\omega)|$, which is decreasing at a rate of $\frac{1}{\omega}$. As a result, the response will be attenuated (i.e., $f(t)$ will present insignificant effects to the response $x(t)$).

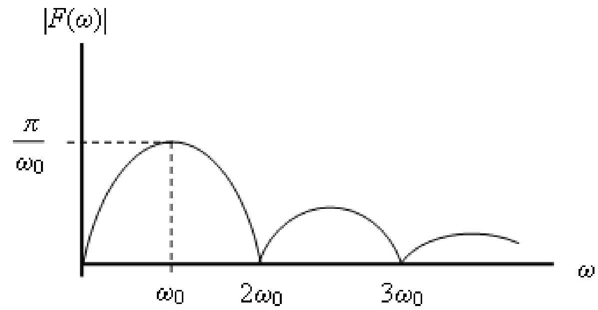


Figure 9: Magnitude of $F(\omega)$.

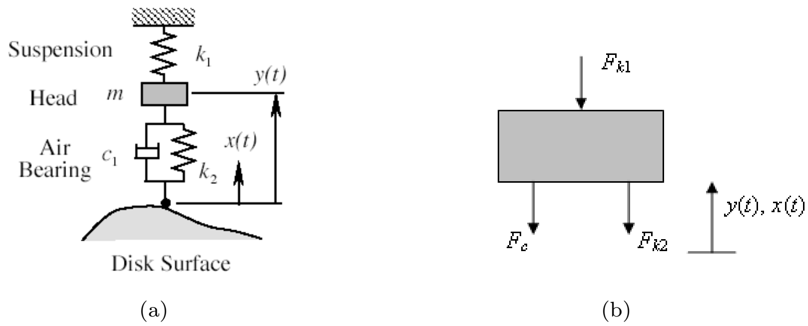


Figure 10: (a) Suspension model in HDD; (b) Free-body diagram of recording head.

Problem 4

(a) From the model for the recording head suspension system illustrated in Figure 10(a), note that $y(t)$ is the relative displacement of the head to the disk surface. Then, from the free-body diagram shown in Figure 10(b), the equation of motion can be determined from

$$\sum F_y = -F_{k_1} - F_{k_2} - F_c = m(\ddot{x} + \ddot{y})$$
$$-k_1(x + y) - k_2 y - c\dot{y} = m(\ddot{x} + \ddot{y})$$

or

$$\underline{\underline{m\ddot{y} + c\dot{y} + (k_1 + k_2)y = -m\ddot{x} - k_1 x.}} \quad (Ans.)$$

(b) The frequency response function $G(\omega)$ can be written as

$$\underline{\underline{G(\omega) = \frac{m\omega^2 - k_1}{(k_1 + k_2) - m\omega^2 + jc\omega}.}} \quad (Ans.)$$

By defining quantities

$$\omega_1 = \sqrt{\frac{k_1}{m}}, \quad \omega_2 = \sqrt{\frac{k_1 + k_2}{m}},$$

$G(\omega)$ can be written as

$$G(\omega) = \frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega^2 + j2\zeta\omega_2\omega}, \qquad \left(\zeta = \frac{c}{2m\omega_2}\right).$$

Then, the magnitude of $G(\omega)$ is

$$|G(\omega)| = \frac{\left|\omega^2 - \omega_1^2\right|}{\sqrt{\left(\omega_2^2 - \omega^2\right)^2 + 4\zeta^2\omega_2^2\omega^2}}. \quad (Ans.)$$

For ease of plotting, we can think of writing the magnitude as $|G(\omega)| = |G_1(\omega)| \cdot |G_2(\omega)|$, where

$$|G_1(\omega)| = \left|\omega^2 - \omega_1^2\right|, \quad |G_2(\omega)| = \frac{1}{\sqrt{\left(\omega_2^2 - \omega^2\right)^2 + 4\zeta^2\omega_2^2\omega^2}}.$$

The plots of $|G_1(\omega)|$ and $|G_2(\omega)|$ are shown in Figure 11, and the plot of $|G(\omega)|$ is shown in Figure 12.

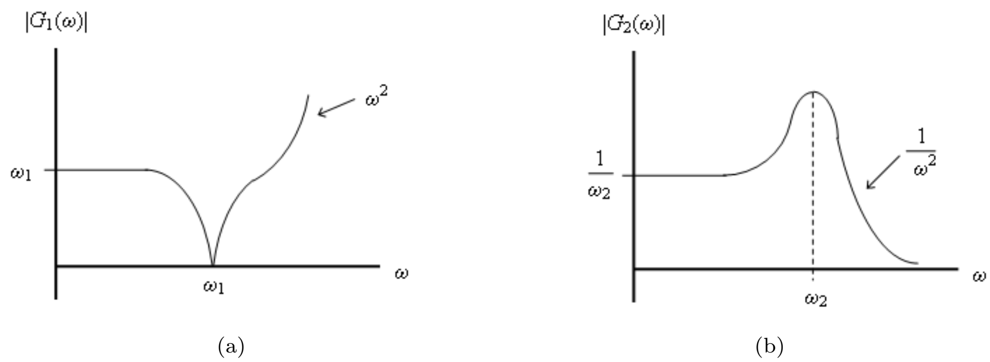


Figure 11: (a) Amplitude of $G_1(\omega)$; (b) Amplitude of $G_2(\omega)$.

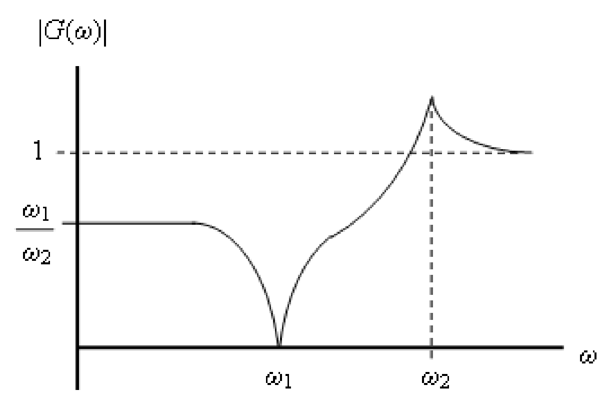


Figure 12: Amplitude of $G(\omega)$.

The phase of $G(\omega)$ is

$$\begin{aligned}\angle G(\omega) &= \angle\left\{\frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega^2 + j2\zeta\omega_2\omega}\right\} \\ &= \angle\left\{\omega^2 - \omega_1^2\right\} - \angle\left\{\omega_2^2 - \omega^2 + j2\zeta\omega_2\omega\right\}\end{aligned}$$

or

$$\angle G(\omega) = \arctan\left(\frac{0}{\omega^2 - \omega_1^2}\right) - \arctan\left(\frac{2\zeta\omega_2\omega}{\omega_2^2 - \omega^2}\right). \quad (Ans.)$$

When $\omega < \omega_1 \ (< \omega_2)$,

$$\arctan\left(\frac{0}{\omega^2 - \omega_1^2}\right) = \pi \quad (\text{or } -\pi), \quad \arctan\left(\frac{2\zeta\omega_2\omega}{\omega_2^2 - \omega^2}\right) = 0.$$

Therefore,

$$\angle G(\omega) = \pi - 0 = \pi \ (\text{or } -\pi) \quad (\text{i.e., displacement of the head is } \textbf{out of phase} \text{ with the disk surface}).$$

When $\omega_1 < \omega < \omega_2$,

$$\arctan\left(\frac{0}{\omega^2 - \omega_1^2}\right) = 0, \quad \arctan\left(\frac{2\zeta\omega_2\omega}{\omega_2^2 - \omega^2}\right) \approx 0.$$

Therefore,

$$\angle G(\omega) = 0 - 0 = 0 \quad (\text{i.e., displacement of the head is } \textbf{in phase} \text{ with the disk surface}).$$

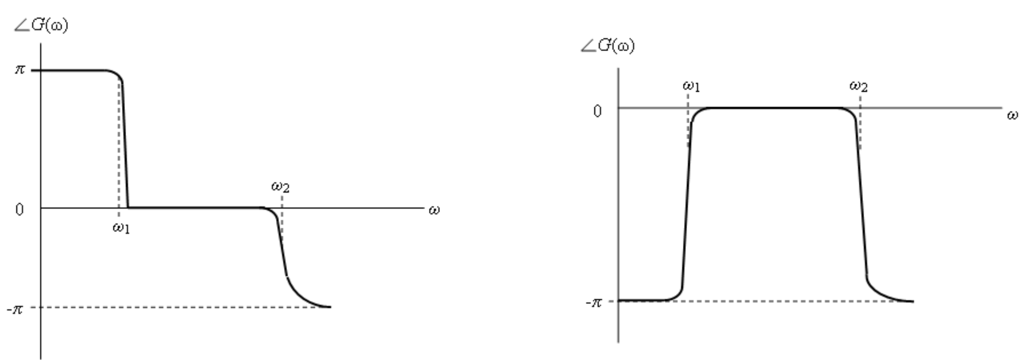
When $\omega > \omega_2$,

$$\arctan\left(\frac{0}{\omega^2 - \omega_1^2}\right) = 0, \quad \arctan\left(\frac{2\zeta\omega_2\omega}{\omega_2^2 - \omega^2}\right) = -\pi.$$

Therefore,

$$\angle G(\omega) = 0 - \pi = -\pi \quad (\text{i.e., displacement of the head is } \textbf{out of phase} \text{ with the disk surface}).$$

The plot of the phase of $G(\omega)$ is shown in Figure 13.



(a) (b)

Figure 13: (a) Phase of $G(\omega)$; (b) Alternative phase plot where $\angle G(\omega \ll \omega_1) = -\pi$.

(c) The recording head follows the disk surface (is in phase with the disk surface) when $\omega_1 < \omega < \omega_2$. The width of this frequency range can be increased by setting $\omega_1 \ll \omega_2$, that is

$$\frac{\omega_1}{\omega_2} \ll 1 \quad \Rightarrow \quad \sqrt{\frac{k_1/m}{(k_1 + k_2)/m}} = \sqrt{\frac{k_1}{k_1 + k_2}} \ll 1 \quad \Rightarrow k_1 \ll k_2.$$

Alternatively, from Figure 12, we see that the magnitude of $G(\omega)$ is constant (and $|G(\omega)| < 1$) for $\omega \ll \omega_1$. Therefore, $|(Y(\omega)| < |X(\omega)|$ (i.e., the displacement between the recording head and the disk surface is small). To increase the bandwidth using this approach, we would want to make $\omega_1 = \sqrt{\frac{k_1}{m}}$ as large as possible.

(d) With the bump on the disk surface, the Fourier transform of

$$x(t) = \begin{cases} h\sin(\pi t/T), & 0 < t < T \\ 0, & t > T \end{cases}$$

is

$$
\begin{aligned}
X(\omega) = \{x(t)\} &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_0^T h \sin\left(\frac{\pi t}{T}\right) e^{-j\omega t}dt \\
&= \int_0^T \frac{h}{2j}\left(e^{j\pi t/T} - e^{-j\pi t/T}\right) e^{-j\omega t}dt \\
&= \frac{h}{2j}\int_0^T \left(e^{j(\pi/T-\omega)t} - e^{-j(\pi/T+\omega)t}\right) dt \\
&= \frac{h}{2j}\left(\frac{1}{j(\pi/T-\omega)}e^{j(\pi/T-\omega)t} + \frac{1}{j(\pi/T+\omega)}e^{-j(\pi/T+\omega)t}\right)\Bigg|_0^T \\
&= -\frac{h}{2(\pi/T-\omega)}\left(e^{j(\pi/T-\omega)T} - 1\right) - \frac{h}{2(\pi/T+\omega)}\left(e^{-j(\pi/T-\omega)T} - 1\right) \\
&= -\frac{h}{2(\pi/T-\omega)}\left(e^{j\pi}e^{-j\omega T} - 1\right) - \frac{h}{2(\pi/T+\omega)}\left(e^{-j\pi}e^{-j\omega T} - 1\right) \\
&= \frac{h}{2(\pi/T-\omega)}\left(e^{-j\omega T} + 1\right) + \frac{h}{2(\pi/T+\omega)}\left(e^{-j\omega T} + 1\right)
\end{aligned}
$$

or

$$
X(\omega) = \frac{h}{\left(\frac{\pi}{T}\right)^2 - \omega^2}\left(e^{-j\omega T} + 1\right). \quad (Ans.)
$$

To find the magnitude, first rewrite $X(\omega)$ as

$$
X(\omega) = \frac{h}{\left(\frac{\pi}{T}\right)^2 - \omega^2}\left(\cos(\omega T) - j\sin(\omega T) + 1\right).
$$

Then,

$$
\begin{aligned}
|X(\omega)| &= \frac{h}{\left|\left(\frac{\pi}{T}\right)^2 - \omega^2\right|}\sqrt{\left(\cos(\omega T) + 1\right)^2 + \sin^2(\omega T)} \\
|X(\omega)| &= \frac{h}{\left|\left(\frac{\pi}{T}\right)^2 - \omega^2\right|}\sqrt{2\cos(\omega T) + 2}.
\end{aligned}
$$

Note that $|X(\omega)| = 0$ when $\omega = \frac{3\pi}{T}, \frac{5\pi}{T}, \frac{7\pi}{T}$, etc. The plot of the amplitude of $X(\omega)$ is shown in Figure 14.

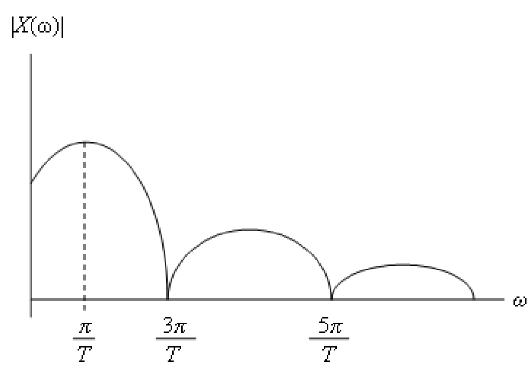


Figure 14: Amplitude of $X(\omega)$.

(e) Recall from (c) that $|G(\omega)|$ remains constant when $\omega \ll \omega_1$. Also, note that $|Y(\omega)| = |G(\omega)| \cdot |X(\omega)|$. Therefore, the minimum T the disk can have without significantly exciting the head into large vibrations can be determined from

$$\omega = \frac{3\pi}{T} \ll \omega_1 \quad \Rightarrow \quad \underline{\underline{T > \frac{3\pi}{\omega_1}}}. \quad (Ans.)$$