$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \qquad (4.6)$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \tag{4.8}$$

$$\varepsilon_{xx} = \left(\frac{1}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yx}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zx}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,xx}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,xx}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,xx}}{G_{xy}}\right)\tau_{xy} \qquad (4.9a)$$

$$\varepsilon_{xx} = \left(\frac{1}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yx}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zx}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,xx}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,xx}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,xx}}{G_{xy}}\right)\tau_{xy} \quad (4.9a)$$

$$\varepsilon_{yy} = \left(\frac{-\nu_{xy}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{1}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zy}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,yy}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,yy}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,yy}}{G_{xy}}\right)\tau_{xy} \quad (4.9b)$$

$$\varepsilon_{xx} = \left(\frac{1}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yx}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zx}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,xx}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,xx}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,xx}}{G_{xy}}\right)\tau_{xy} \quad (4.9a)$$

$$\varepsilon_{yy} = \left(\frac{-\nu_{xy}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{1}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zy}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,yy}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,yy}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,yy}}{G_{xy}}\right)\tau_{xy} \quad (4.9b)$$

$$\varepsilon_{zz} = \left(\frac{-\nu_{xz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{1}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,zz}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,zz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,zz}}{G_{xy}}\right)\tau_{xy} \quad (4.9c)$$

$$\varepsilon_{xx} = \left(\frac{1}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yx}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zx}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,xx}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,xx}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,xx}}{G_{xy}}\right)\tau_{xy} \tag{4.9a}$$

$$\varepsilon_{yy} = \left(\frac{-\nu_{xy}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{1}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zy}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,yy}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,yy}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,yy}}{G_{xy}}\right)\tau_{xy} \tag{4.9b}$$

$$\varepsilon_{zz} = \left(\frac{-\nu_{xz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{1}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,zz}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,zz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,zz}}{G_{xy}}\right)\tau_{xy} \tag{4.9c}$$

$$\gamma_{yz} = \left(\frac{\eta_{xx,yz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{\eta_{yy,yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{\eta_{zz,yz}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{1}{G_{yz}}\right)\tau_{yz} + \left(\frac{\mu_{xz,yz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\mu_{xy,yz}}{G_{xy}}\right)\tau_{xy} \tag{4.9d}$$

$$\varepsilon_{xx} = \left(\frac{1}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yx}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zx}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,xx}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,xx}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,xx}}{G_{xy}}\right)\tau_{xy} \qquad (4.9a)$$

$$\varepsilon_{yy} = \left(\frac{-\nu_{xy}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{1}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zy}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,yy}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,yy}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,yy}}{G_{xy}}\right)\tau_{xy} \qquad (4.9b)$$

$$\varepsilon_{zz} = \left(\frac{-\nu_{xz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{1}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,zz}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,zz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,zz}}{G_{xy}}\right)\tau_{xy} \qquad (4.9c)$$

$$\gamma_{yz} = \left(\frac{\eta_{xx,yz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{\eta_{yy,yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{\eta_{zz,yz}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{1}{G_{yz}}\right)\tau_{yz} + \left(\frac{\mu_{xz,yz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\mu_{xy,yz}}{G_{xy}}\right)\tau_{xy} \qquad (4.9d)$$

$$\gamma_{xz} = \left(\frac{\eta_{xx,xz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{\eta_{yy,xz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{\eta_{zz,xz}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\mu_{yz,xz}}{G_{yz}}\right)\tau_{yz} + \left(\frac{1}{G_{xz}}\right)\tau_{xz} + \left(\frac{\mu_{xy,xz}}{G_{xy}}\right)\tau_{xy} \qquad (4.9e)$$

$$\varepsilon_{xx} = \left(\frac{1}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yx}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zx}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,xx}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,xx}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,xx}}{G_{xy}}\right)\tau_{xy} \quad (4.9a)$$

$$\varepsilon_{yy} = \left(\frac{-\nu_{xy}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{1}{E_{yy}}\right)\sigma_{yy} + \left(\frac{-\nu_{zy}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,yy}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,yy}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,yy}}{G_{xy}}\right)\tau_{xy} \quad (4.9b)$$

$$\varepsilon_{zz} = \left(\frac{-\nu_{xz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{-\nu_{yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{1}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\eta_{yz,zz}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\eta_{xz,zz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\eta_{xy,zz}}{G_{xy}}\right)\tau_{xy} \quad (4.9c)$$

$$\gamma_{yz} = \left(\frac{\eta_{xx,yz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{\eta_{yy,yz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{\eta_{zz,yz}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{1}{G_{yz}}\right)\tau_{yz} + \left(\frac{\mu_{xz,yz}}{G_{xz}}\right)\tau_{xz} + \left(\frac{\mu_{xy,yz}}{G_{xy}}\right)\tau_{xy} \quad (4.9d)$$

$$\gamma_{xz} = \left(\frac{\eta_{xx,xz}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{\eta_{yy,xz}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{\eta_{zz,xz}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\mu_{yz,xz}}{G_{yz}}\right)\tau_{yz} + \left(\frac{1}{G_{xz}}\right)\tau_{xz} + \left(\frac{\mu_{xy,xz}}{G_{xy}}\right)\tau_{xy} \quad (4.9e)$$

$$\gamma_{xy} = \left(\frac{\eta_{xx,xy}}{E_{xx}}\right)\sigma_{xx} + \left(\frac{\eta_{yy,xy}}{E_{yy}}\right)\sigma_{yy} + \left(\frac{\eta_{zz,xy}}{E_{zz}}\right)\sigma_{zz} + \left(\frac{\mu_{yz,xy}}{G_{yz}}\right)\tau_{yz} + \left(\frac{\mu_{xz,xy}}{G_{xz}}\right)\tau_{xz} + \left(\frac{1}{G_{xy}}\right)\tau_{xy} \quad (4.9f)$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \left(\frac{1}{E_{xx}}\right) & \left(\frac{-\nu_{yx}}{E_{yy}}\right) & \left(\frac{-\nu_{zx}}{E_{zz}}\right) & \left(\frac{\eta_{yz,xx}}{G_{yz}}\right) & \left(\frac{\eta_{xz,xx}}{G_{xz}}\right) & \left(\frac{\eta_{xy,xx}}{G_{xy}}\right) \\ \left(\frac{-\nu_{xy}}{E_{xx}}\right) & \left(\frac{1}{E_{yy}}\right) & \left(\frac{-\nu_{zy}}{E_{zz}}\right) & \left(\frac{\eta_{yz,yy}}{G_{yz}}\right) & \left(\frac{\eta_{xz,yy}}{G_{xz}}\right) & \left(\frac{\eta_{xy,yy}}{G_{xy}}\right) \\ \left(\frac{-\nu_{xz}}{E_{xx}}\right) & \left(\frac{-\nu_{yz}}{E_{yy}}\right) & \left(\frac{1}{E_{zz}}\right) & \left(\frac{\eta_{yz,zz}}{G_{yz}}\right) & \left(\frac{\eta_{xz,zz}}{G_{xz}}\right) & \left(\frac{\eta_{xy,zz}}{G_{xy}}\right) \\ \left(\frac{\eta_{xx,yz}}{E_{xx}}\right) & \left(\frac{\eta_{yy,yz}}{E_{yy}}\right) & \left(\frac{\eta_{zz,yz}}{E_{zz}}\right) & \left(\frac{1}{G_{yz}}\right) & \left(\frac{\mu_{xz,yz}}{G_{xz}}\right) & \left(\frac{\mu_{xy,yz}}{G_{xy}}\right) \\ \left(\frac{\eta_{xx,xz}}{E_{xx}}\right) & \left(\frac{\eta_{yy,xz}}{E_{yy}}\right) & \left(\frac{\eta_{zz,xz}}{E_{zz}}\right) & \left(\frac{\mu_{yz,xz}}{G_{yz}}\right) & \left(\frac{1}{G_{xz}}\right) & \left(\frac{\mu_{xy,xz}}{G_{xy}}\right) \\ \left(\frac{\eta_{xx,xy}}{E_{xx}}\right) & \left(\frac{\eta_{yy,xy}}{E_{yy}}\right) & \left(\frac{\eta_{zz,xy}}{E_{zz}}\right) & \left(\frac{\mu_{yz,xy}}{G_{yz}}\right) & \left(\frac{\mu_{xz,xy}}{G_{xz}}\right) & \left(\frac{1}{G_{xy}}\right) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \qquad (4.10)$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \tag{4.8}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \left(\frac{1}{E_{xx}}\right) & \left(\frac{-\nu_{yx}}{E_{yy}}\right) & \left(\frac{-\nu_{zx}}{E_{zz}}\right) & \left(\frac{\eta_{yz,xx}}{G_{yz}}\right) & \left(\frac{\eta_{xz,xx}}{G_{xz}}\right) & \left(\frac{\eta_{xy,xx}}{G_{xy}}\right) \\ \left(\frac{-\nu_{xy}}{E_{xx}}\right) & \left(\frac{1}{E_{yy}}\right) & \left(\frac{-\nu_{zy}}{E_{zz}}\right) & \left(\frac{\eta_{yz,yy}}{G_{yz}}\right) & \left(\frac{\eta_{xz,yy}}{G_{xz}}\right) & \left(\frac{\eta_{xy,yy}}{G_{xy}}\right) \\ \left(\frac{-\nu_{xz}}{E_{xx}}\right) & \left(\frac{-\nu_{yz}}{E_{yy}}\right) & \left(\frac{1}{E_{zz}}\right) & \left(\frac{\eta_{yz,zz}}{G_{yz}}\right) & \left(\frac{\eta_{xz,zz}}{G_{xz}}\right) & \left(\frac{\eta_{xy,zz}}{G_{xy}}\right) \\ \left(\frac{\eta_{xx,yz}}{E_{xx}}\right) & \left(\frac{\eta_{yy,yz}}{E_{yy}}\right) & \left(\frac{\eta_{zz,yz}}{E_{zz}}\right) & \left(\frac{1}{G_{yz}}\right) & \left(\frac{\mu_{xz,yz}}{G_{xz}}\right) & \left(\frac{\mu_{xy,yz}}{G_{xy}}\right) \\ \left(\frac{\eta_{xx,xz}}{E_{xx}}\right) & \left(\frac{\eta_{yy,xz}}{E_{yy}}\right) & \left(\frac{\eta_{zz,xz}}{E_{zz}}\right) & \left(\frac{\mu_{yz,xz}}{G_{yz}}\right) & \left(\frac{1}{G_{xz}}\right) & \left(\frac{\mu_{xy,xz}}{G_{xy}}\right) \\ \left(\frac{\eta_{xx,xy}}{E_{xx}}\right) & \left(\frac{\eta_{yy,xy}}{E_{yy}}\right) & \left(\frac{\eta_{zz,xy}}{E_{zz}}\right) & \left(\frac{\mu_{yz,xy}}{G_{yz}}\right) & \left(\frac{\mu_{xz,xy}}{G_{xz}}\right) & \left(\frac{1}{G_{xy}}\right) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \qquad (4.10)$$

By inspection : $S_{11} = \dfrac{1}{E_{11}}, \; S_{12} = \dfrac{-\nu_{yx}}{E_{yy}}, \; \ldots, \; S_{16} = \dfrac{\eta_{xy,xx}}{G_{xy}}$, etc

- " It can be shown" that the compliance matrix S_{ij} must *also* be symmetric:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \qquad (4.8)$$

$$\begin{array}{lllll} S_{21} = S_{12} & & & & \\ S_{31} = S_{13} & S_{32} = S_{23} & & & \\ S_{41} = S_{14} & S_{42} = S_{24} & S_{43} = S_{34} & & \\ S_{51} = S_{15} & S_{52} = S_{25} & S_{53} = S_{35} & S_{54} = S_{45} & \\ S_{61} = S_{16} & S_{62} = S_{26} & S_{63} = S_{36} & S_{64} = S_{46} & S_{65} = S_{56} \end{array} \qquad (4.7)$$

Number of material constants:

- The compliance matrix, S_{ijkl}, is a 4th-order tensor → 81 constants (apparently)

Number of material constants:

- The compliance matrix, S_{ijkl}, is a 4th-order tensor → 81 constants (apparently)
- The fact that $[\sigma_{ij}]$ and $[\varepsilon_{ij}]$ are symmetric allowed us to adopt "contracted notation" (and therefore to adopt matrix notation):

$$S_{ijkl} \rightarrow S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \rightarrow \text{ 36 constants (apparently)}$$

Summary:

- "It can be shown" that the compliance matrix *must also be symmetric*:

$$S_{ij}=\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix}=\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix}\rightarrow 21 \text{ constants}$$

- An anisotropic material has 21 independent material constants

- " It can be shown" that the compliance matrix S_{ij} must also be symmetric:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \qquad (4.8)$$

$$\begin{aligned} &S_{21} = S_{12} \\ &S_{31} = S_{13} \quad S_{32} = S_{23} \\ &S_{41} = S_{14} \quad S_{42} = S_{24} \quad S_{43} = S_{34} \\ &S_{51} = S_{15} \quad S_{52} = S_{25} \quad S_{53} = S_{35} \quad S_{54} = S_{45} \\ &S_{61} = S_{16} \quad S_{62} = S_{26} \quad S_{63} = S_{36} \quad S_{64} = S_{46} \quad S_{65} = S_{56} \end{aligned} \qquad (4.7)$$

$$S_{11} = \frac{1}{E_{xx}} \qquad S_{22} = \frac{1}{E_{yy}} \qquad S_{33} = \frac{1}{E_{zz}}$$

$$S_{44} = \frac{1}{G_{yz}} \qquad S_{55} = \frac{1}{G_{xz}} \qquad S_{66} = \frac{1}{G_{xy}}$$

$$S_{21} = S_{12} = \frac{-\nu_{xy}}{E_{xx}} = \frac{-\nu_{yx}}{E_{yy}} \qquad S_{31} = S_{13} = \frac{-\nu_{xz}}{E_{xx}} = \frac{-\nu_{zx}}{E_{zz}}$$

$$S_{32} = S_{23} = \frac{-\nu_{yz}}{E_{yy}} = \frac{-\nu_{zy}}{E_{zz}} \qquad S_{41} = S_{14} = \frac{\eta_{xx,yz}}{E_{xx}} = \frac{\eta_{yz,xx}}{G_{yz}}$$

$$S_{42} = S_{24} = \frac{\eta_{yy,yz}}{E_{yy}} = \frac{\eta_{yz,yy}}{G_{yz}} \qquad S_{43} = S_{34} = \frac{\eta_{zz,yz}}{E_{zz}} = \frac{\eta_{yz,zz}}{G_{yz}}$$

$$S_{51} = S_{15} = \frac{\eta_{xx,xz}}{E_{xx}} = \frac{\eta_{xz,xx}}{G_{xz}} \qquad S_{52} = S_{25} = \frac{\eta_{yy,xz}}{E_{yy}} = \frac{\eta_{xz,yy}}{G_{xz}} \qquad (4.11)$$

$$S_{53} = S_{35} = \frac{\eta_{zz,xz}}{E_{zz}} = \frac{\eta_{xz,zz}}{G_{xz}} \qquad S_{54} = S_{45} = \frac{\mu_{yz,xz}}{G_{yz}} = \frac{\mu_{xz,yz}}{G_{xz}}$$

$$S_{61} = S_{16} = \frac{\eta_{xx,xy}}{E_{xx}} = \frac{\eta_{xy,xx}}{G_{xy}} \qquad S_{62} = S_{26} = \frac{\eta_{yy,xy}}{E_{yy}} = \frac{\eta_{xy,yy}}{G_{xy}}$$

$$S_{63} = S_{36} = \frac{\eta_{zz,xy}}{E_{zz}} = \frac{\eta_{xy,zz}}{G_{xy}} \qquad S_{64} = S_{46} = \frac{\mu_{yz,xy}}{G_{yz}} = \frac{\mu_{xy,yz}}{G_{xy}}$$

$$S_{65} = S_{56} = \frac{\mu_{xz,xy}}{G_{xz}} = \frac{\mu_{xy,xz}}{G_{xy}}$$

…symmetry of the compliance matrix leads to the "inverse relations"

$$\frac{\nu_{xy}}{E_{xx}} = \frac{\nu_{yx}}{E_{yy}}$$

$$\frac{\nu_{xz}}{E_{xx}} = \frac{\nu_{zx}}{E_{zz}} \qquad \frac{\nu_{yz}}{E_{yy}} = \frac{\nu_{zy}}{E_{zz}}$$

$$\frac{\eta_{xx,yz}}{E_{xx}} = \frac{\eta_{yz,xx}}{G_{yz}} \qquad \frac{\eta_{yy,yz}}{E_{yy}} = \frac{\eta_{yz,yy}}{G_{yz}} \qquad \frac{\eta_{zz,yz}}{E_{zz}} = \frac{\eta_{yz,zz}}{G_{yz}} \qquad (4.12)$$

$$\frac{\eta_{xx,xz}}{E_{xx}} = \frac{\eta_{xz,xx}}{G_{xz}} \qquad \frac{\eta_{yy,xz}}{E_{yy}} = \frac{\eta_{xz,yy}}{G_{xz}} \qquad \frac{\eta_{zz,xz}}{E_{zz}} = \frac{\eta_{xz,zz}}{G_{xz}} \qquad \frac{\mu_{yz,xz}}{G_{yz}} = \frac{\mu_{xz,yz}}{G_{xz}}$$

$$\frac{\eta_{xx,xy}}{E_{xx}} = \frac{\eta_{xy,xx}}{G_{xy}} \qquad \frac{\eta_{yy,xy}}{E_{yy}} = \frac{\eta_{xy,yy}}{G_{xy}} \qquad \frac{\eta_{zz,xy}}{E_{zz}} = \frac{\eta_{xy,zz}}{G_{xy}} \qquad \frac{\mu_{yz,xy}}{G_{yz}} = \frac{\mu_{xy,yz}}{G_{xy}} \qquad \frac{\mu_{xz,xy}}{G_{xz}} = \frac{\mu_{xy,xz}}{G_{xy}}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{4.14}$$

Where $C_{ij} = S_{ij}^{-1}$

Orthotropic materials

(when referenced to the Principal Material Coordinate System)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \left(\frac{1}{E_{11}}\right) & \left(\frac{-\nu_{21}}{E_{22}}\right) & \left(\frac{-\nu_{31}}{E_{33}}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{12}}{E_{11}}\right) & \left(\frac{1}{E_{22}}\right) & \left(\frac{-\nu_{32}}{E_{33}}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{13}}{E_{11}}\right) & \left(\frac{-\nu_{23}}{E_{22}}\right) & \left(\frac{1}{E_{33}}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1}{G_{23}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1}{G_{13}}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{1}{G_{12}}\right) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (4.15)$$

- An orthotropic material has 9 independent material constants….

Orthotropic materials

(when referenced to the Principal Material Coordinate System)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (4.16)$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \qquad (4.18)$$

Transversely Isotropic Materials
(when referenced to the Principal Material Coordinate System)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \left(\frac{1}{E_{11}}\right) & \left(\frac{-\nu_{21}}{E_{22}}\right) & \left(\frac{-\nu_{21}}{E_{22}}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{12}}{E_{11}}\right) & \left(\frac{1}{E_{22}}\right) & \left(\frac{-\nu_{23}}{E_{22}}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{12}}{E_{11}}\right) & \left(\frac{-\nu_{23}}{E_{22}}\right) & \left(\frac{1}{E_{22}}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{2(1+\nu_{23})}{E_{22}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1}{G_{12}}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{1}{G_{12}}\right) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Eq (4.21)

- A transversely isotropic material has 5 independent material constants….

Transversely Isotropic materials

(when referenced to the Principal Material Coordinate System)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (4.22)$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{22}-C_{23})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \qquad (4.23)$$

Hooke's Law: Anisotropic Materials

(including Orthotropic and Transversely Isotropic materials, when referenced to a non-principal coordinate system)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{zz} \\ \alpha_{yz} \\ \alpha_{xz} \\ \alpha_{xy} \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{xx} \\ \beta_{yy} \\ \beta_{zz} \\ \beta_{yz} \\ \beta_{xz} \\ \beta_{xy} \end{Bmatrix} \qquad \textbf{(4.30)}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy} \\ \varepsilon_{zz} - \Delta T \alpha_{zz} - \Delta M \beta_{zz} \\ \gamma_{yz} - \Delta T \alpha_{yz} - \Delta M \beta_{yz} \\ \gamma_{xz} - \Delta T \alpha_{xz} - \Delta M \beta_{xz} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy} \end{Bmatrix} \qquad \textbf{(4.31)}$$

Hooke's Law: Orthotropic Materials

(when referenced to the Principal Material Coordinate System)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \textbf{(4.32)}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} - \Delta T \alpha_{11} - \Delta M \beta_{11} \\ \varepsilon_{22} - \Delta T \alpha_{22} - \Delta M \beta_{22} \\ \varepsilon_{33} - \Delta T \alpha_{33} - \Delta M \beta_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \qquad \textbf{(4.33)}$$

Hooke's Law: Transversely Isotropic Materials
(when referenced to the Principal Material Coordinate System)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \textbf{(4.34)}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{22}-C_{23})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} - \Delta T\alpha_{11} - \Delta M\beta_{11} \\ \varepsilon_{22} - \Delta T\alpha_{22} - \Delta M\beta_{22} \\ \varepsilon_{33} - \Delta T\alpha_{22} - \Delta M\beta_{22} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \quad \textbf{(4.35)}$$