ME450

Topics Covered During Quarter

- Reading Assignments: Chapters 1 through 7
  - Invariant analyses will not be covered on final exam (section 5.3 and 6.10)
  - Final exam will be comprehensive, but
  - Will focus on topics discussed since the midterm

- Final exam format (nominal times):
  - Multiple choice portion administered via Catalyst system from 12:30pm-1:20pm Friday 9 March
  - Written problems sent via e-mail at 5pm Friday 9 March , upload solutions to Canvas dropbox by 5pm Saturday 10 March

ME450

Topics Covered During Quarter

- Chapter 1: Introduction
  - 3 classes of composites: PMCs, MMCs, CMCs
  - Polymers: basic concepts (molecular structures, thermoplastic vs thermosets, T_g, etc)
  - Advanced Fibers:
    - Major types (glass, aramid, carbon, UHDPE)
    - Commercial forms (discontinuous, roving, woven, braided, prepreg)
  - Manufacturing
    - “Dry” vs “wet”
    - Hand layup, filament winding, automated tape laying, automated fiber placement, resin infusion, autoclaves, chopped fiber sprayup, compression molding, injection molding, pultrusion, ….

ME450

Topics Covered During Quarter

- Chapter 2: Review of Force Stress and Strain Tensors
  - Force, stress, and strain tensors
  - 3-D transformation of a tensor: direction cosines, c_{ij}

$$F_{i'} = c_{i'j} F_j \qquad \sigma_{i'j'} = c_{i'k} c_{j'l} \sigma_{kl} \qquad [\varepsilon_{i'j'}] = [c_{i'j}][\varepsilon_{ij}][c_{i'j}]^T$$

  - Stress or strain transformation within a plane:

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

ME450

Topics Covered During Quarter

- Chapter 3: Material Properties
  - Elastic properties measured during
    - uniaxial stress tests (Young's modulus, Poisson ratios, coefficients of mutual influence 2nd kind) → 18 properties
    - pure shear tests (Shear modulus, Chentsov coefficients, coefficients of mutual influence 2nd kind) → 18 properties
  - Coefficients of thermal and moisture expansion (α's and β's)
  - Principal material coordinate system (μ's, η's $= 0$)
  - Failure strengths measured in principal material coordinate system

ME450

Topics Covered During Quarter

- Chapter 4: Elastic Response of Anisotropic Materials

$$\varepsilon_{ij} = \varepsilon_{ij}^{\sigma} + \varepsilon_{ij}^{T} + \varepsilon_{ij}^{M}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{zz} \\ \alpha_{yz} \\ \alpha_{xz} \\ \alpha_{xy} \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{xx} \\ \beta_{yy} \\ \beta_{zz} \\ \beta_{yz} \\ \beta_{xz} \\ \beta_{xy} \end{Bmatrix}$$

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Topics Covered During Quarter

- Chapter 4: Elastic Response of Anisotropic Materials

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T\alpha_{xx} - \Delta M\beta_{xx} \\ \varepsilon_{yy} - \Delta T\alpha_{yy} - \Delta M\beta_{yy} \\ \varepsilon_{zz} - \Delta T\alpha_{zz} - \Delta M\beta_{zz} \\ \gamma_{yz} - \Delta T\alpha_{yz} - \Delta M\beta_{yz} \\ \gamma_{xz} - \Delta T\alpha_{xz} - \Delta M\beta_{xz} \\ \gamma_{xy} - \Delta T\alpha_{xy} - \Delta M\beta_{xy} \end{Bmatrix}$$

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Topics Covered During Quarter

- Chapter 4: Elastic Response of Anisotropic Materials
  In Principal Material Coordinate System

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Topics Covered During Quarter

- Chapter 4: Elastic Response of Anisotropic Materials
  In Principal Material Coordinate System

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} - \Delta T\alpha_{11} - \Delta M\beta_{11} \\ \varepsilon_{22} - \Delta T\alpha_{22} - \Delta M\beta_{22} \\ \varepsilon_{33} - \Delta T\alpha_{33} - \Delta M\beta_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

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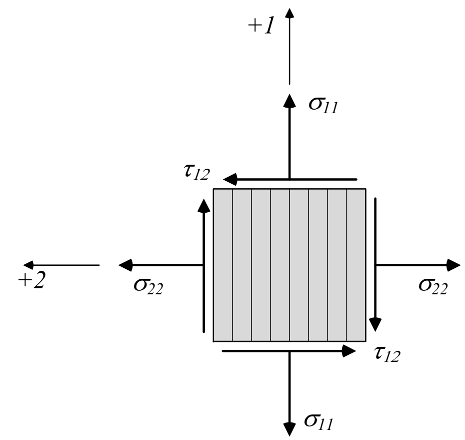
Topics Covered During Quarter

- <u>Chapter 5</u>: Unidirectional Composite Laminates Subject to Plane Stress:

$$\text{Assuming } \sigma_{zz} = \tau_{xz} = \tau_{yz} = 0:$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

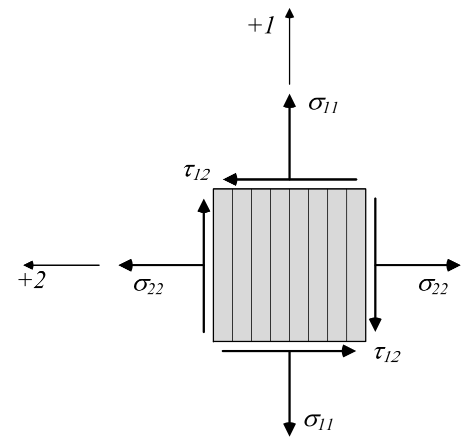
Sec 5.1: Unidirectional composites referenced to principal material coordinate system



$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ 0 \end{Bmatrix}$$

$$\varepsilon_{33} = \varepsilon_{zz} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + \Delta T \alpha_{33} + \Delta M \beta_{33}$$

Sec 5.1: Unidirectional composites referenced to principal material coordinate system



$$S_{11} = \frac{1}{E_{11}} \qquad S_{22} = \frac{1}{E_{22}}$$

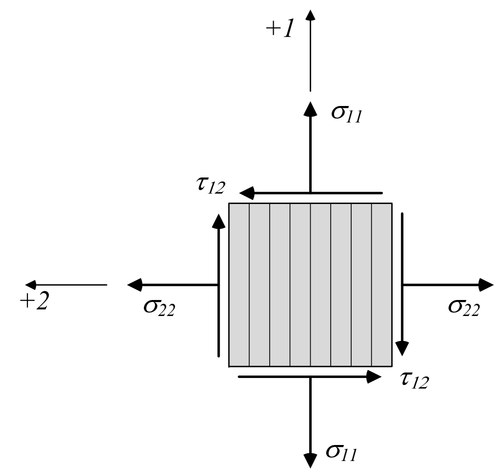
$$S_{21} = S_{12} = \frac{-\nu_{12}}{E_{11}} = \frac{-\nu_{21}}{E_{22}}$$

$$S_{66} = \frac{1}{G_{12}}$$

$$S_{31} = S_{13} = \frac{-\nu_{13}}{E_{11}} = \frac{-\nu_{31}}{E_{33}}$$

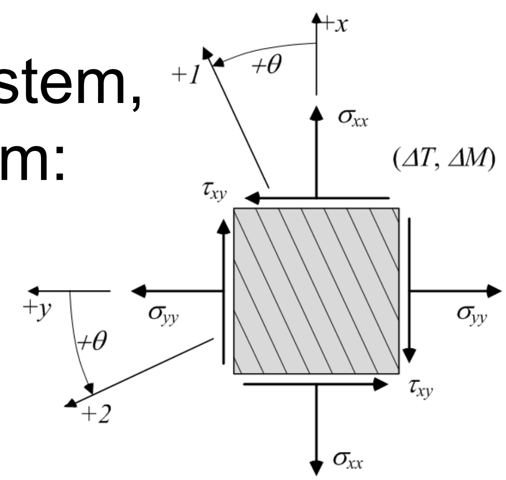
$$S_{32} = S_{23} = \frac{-\nu_{23}}{E_{22}} = \frac{-\nu_{32}}{E_{33}}$$

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system



•Starting with Hooke's law in the *1-2* coordinate system:

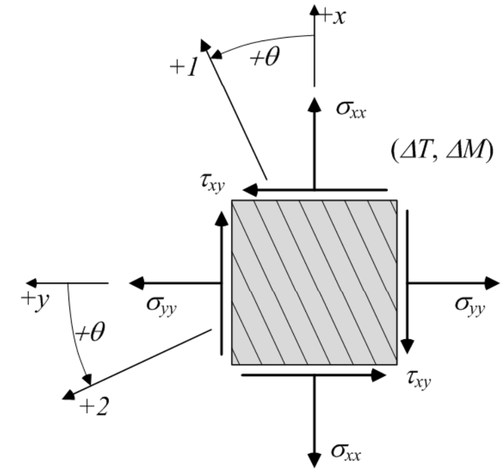
$$\{\varepsilon\}_{1,2} = [S][\sigma]_{1,2} + \Delta T\{\alpha\}_{1,2} + \Delta M\{\beta\}_{1,2}$$



…we rotated to an arbitrary *x-y* coordinate system, oriented θ degs from the *1-2* coordinate system:

$$\{\varepsilon\}_{x,y} = [\bar{S}][\sigma]_{x,y} + \Delta T\{\alpha\}_{x,y} + \Delta M\{\beta\}_{x,y}$$

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system



$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{xx} \\ \beta_{yy} \\ \beta_{xy} \end{Bmatrix}$$

$$\varepsilon_{33} = \varepsilon_{zz} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + \Delta T \alpha_{33} + \Delta M \beta_{33}$$

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system

$$\bar{S}_{11} = S_{11}\cos^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta + S_{22}\sin^4\theta$$

$$\bar{S}_{12} = \bar{S}_{21} = S_{12}(\cos^4\theta + \sin^4\theta) + (S_{11} + S_{22} - S_{66})\cos^2\theta\sin^2\theta$$

$$\bar{S}_{16} = \bar{S}_{61} = (2S_{11} - 2S_{12} - S_{66})\cos^3\theta\sin\theta - (2S_{22} - 2S_{12} - S_{66})\cos\theta\sin^3\theta$$

$$\bar{S}_{22} = S_{11}\sin^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta + S_{22}\cos^4\theta$$

$$\bar{S}_{26} = \bar{S}_{62} = (2S_{11} - 2S_{12} - S_{66})\cos\theta\sin^3\theta - (2S_{22} - 2S_{12} - S_{66})\cos^3\theta\sin\theta$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\cos^2\theta\sin^2\theta + S_{66}(\cos^4\theta + \sin^4\theta)$$

Equations (5.22)

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system

$$\alpha_{xx} = \alpha_{11}\cos^2(\theta) + \alpha_{22}\sin^2(\theta)$$

$$\alpha_{yy} = \alpha_{11}\sin^2(\theta) + \alpha_{22}\cos^2(\theta) \qquad (5.25)$$

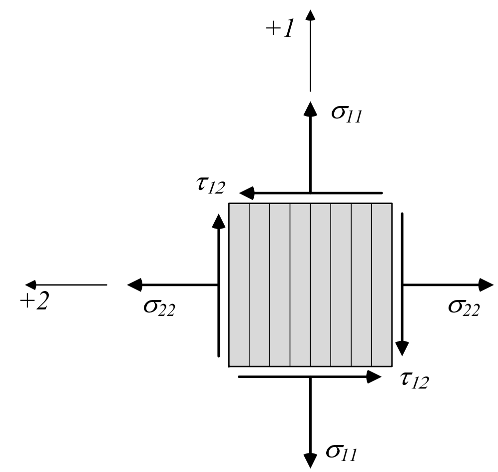
$$\alpha_{xy} = 2\cos(\theta)\sin(\theta)(\alpha_{11} - \alpha_{22})$$

$$\beta_{xx} = \beta_{11}\cos^2(\theta) + \beta_{22}\sin^2(\theta)$$

$$\beta_{yy} = \beta_{11}\sin^2(\theta) + \beta_{22}\cos^2(\theta) \qquad (5.28)$$

$$\beta_{xy} = 2\cos(\theta)\sin(\theta)(\beta_{11} - \beta_{22})$$

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system

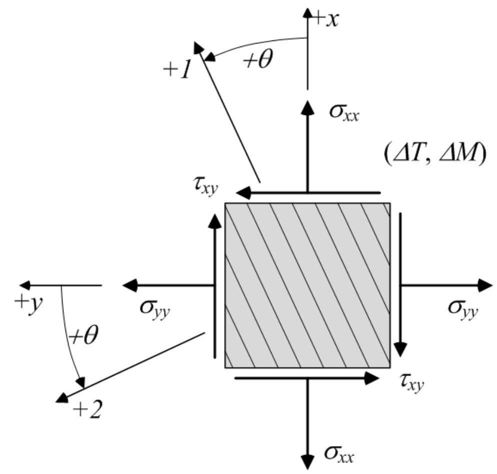


- Similarly:

$$\{\sigma\}_{1,2} = [Q]\{\varepsilon - \Delta T\alpha - \Delta M\beta\}_{1,2}$$

$$\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \equiv \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}^{-1}$$

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system



- Similarly:

$$\{\sigma\}_{x,y} = [\overline{Q}]\{\varepsilon - \Delta T\alpha - \Delta M\beta\}_{x,y}$$

$$[\overline{Q}] = [\overline{S}]^{-1} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}$$

Sec 5.2: Unidirectional composites referenced to an arbitrary coordinate system

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\sin^4\theta$$

$$\overline{Q}_{12} = \overline{Q}_{21} = Q_{12}(\cos^4\theta + \sin^4\theta) + (Q_{11} + Q_{22} - 4Q_{66})\cos^2\theta\sin^2\theta$$

$$\overline{Q}_{16} = \overline{Q}_{61} = (Q_{11} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta$$

$$\overline{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\cos^4\theta$$

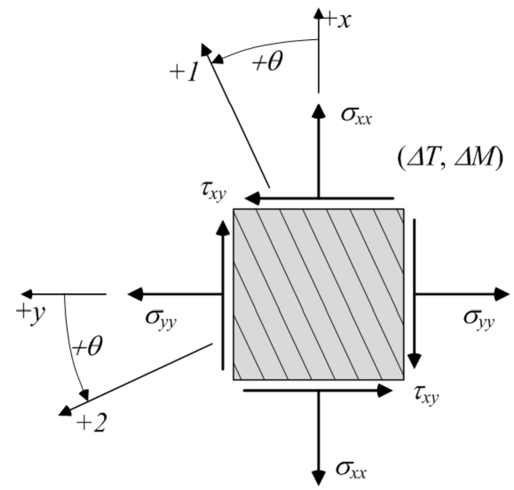
$$\overline{Q}_{26} = \overline{Q}_{62} = (Q_{11} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\cos^2\theta\sin^2\theta + Q_{66}(\cos^4\theta + \sin^4\theta)$$

Equations (5.31)

Sec 5.4: Effective properties of unidirectional laminates

- Hooke's Law for Composites Referenced to an Arbitrary *x-y-z* Coordinate System (Plane Stress)

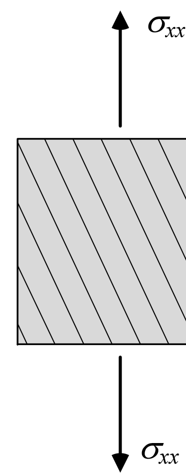


$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16}\\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26}\\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66}\end{bmatrix}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}+\Delta T\begin{Bmatrix}\alpha_{xx}\\ \alpha_{yy}\\ \alpha_{xy}\end{Bmatrix}+\Delta M\begin{Bmatrix}\beta_{xx}\\ \beta_{yy}\\ \beta_{xy}\end{Bmatrix}$$

$$\varepsilon_{33}=\varepsilon_{zz}=S_{13}\sigma_{11}+S_{23}\sigma_{22}+\Delta T\alpha_{33}+\Delta M\beta_{33}$$

Sec 5.4: Effective properties of unidirectional laminates

- Effective properties determined by applying the "normal" definition.
- For example, if only σ_{xx} applied:



$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \bar{S}_{11}\sigma_{xx} \\ \bar{S}_{12}\sigma_{xx} \\ \bar{S}_{16}\sigma_{xx} \end{Bmatrix}$$

$$E_{xx} = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{1}{\bar{S}_{11}} \qquad \nu_{xy} = \frac{-\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-\bar{S}_{12}}{\bar{S}_{11}} \qquad \eta_{xx,xy} = \frac{\gamma_{xy}}{\varepsilon_{xx}} = \frac{\bar{S}_{16}}{\bar{S}_{11}}$$

Sec 5.4: Effective properties of unidirectional laminates

- Similarly:

$$E_{yy} = \frac{1}{\bar{S}_{22}} \qquad \nu_{yx} = \frac{-\bar{S}_{12}}{\bar{S}_{22}} \qquad \eta_{yy,xy} = \frac{\bar{S}_{26}}{\bar{S}_{22}}$$

$$G_{xy} = \frac{1}{\bar{S}_{66}} \qquad \eta_{xy,xx} = \frac{\bar{S}_{16}}{\bar{S}_{66}} \qquad \eta_{xy,yy} = \frac{\bar{S}_{26}}{\bar{S}_{66}}$$

Sec 5.5, 5.6: Macroscopic Failure Theories

- Dozens of failure theories have been proposed…none are universally accepted
- Three common failure criterion described in textbook:
  - Maximum stress failure criterion
  - Tsai-Hill failure criterion
  - Tsai-Wu failure criterion

Maximum Stress Failure Criterion

(Plane stress form)

- Failure does not occur if:

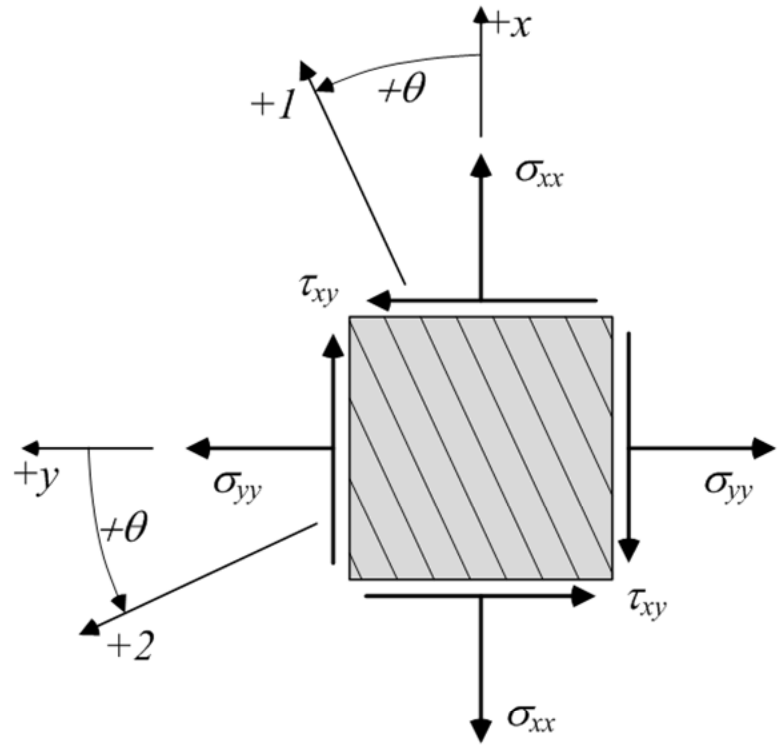
$$-1*\sigma_{11}^{fC} < \sigma_{11} < \sigma_{11}^{fT}$$

(and)

$$-1*\sigma_{22}^{fC} < \sigma_{22} < \sigma_{22}^{fT}$$

(and)

$$|\tau_{12}| < \tau_{12}^{f}$$

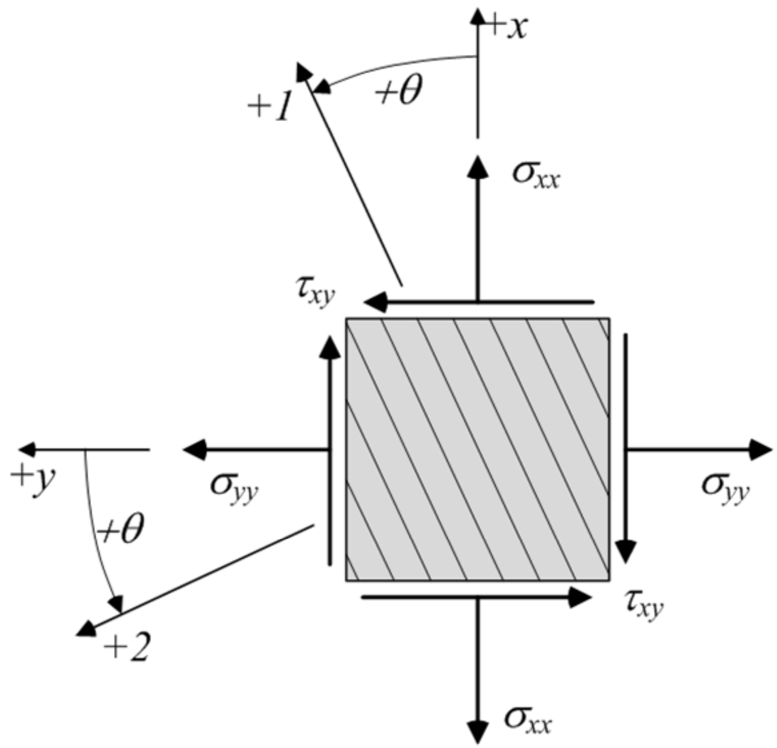


Tsai-Hill Failure Criterion

(Plane stress form)

- Failure does not occur if:

$$\frac{(\sigma_{11})^2}{\left(\sigma_{11}^{fT}\right)^2} + \frac{(\sigma_{22})^2}{\left(\sigma_{22}^{fT}\right)^2} + \frac{(\tau_{12})^2}{\left(\tau_{12}^{f}\right)^2} - \frac{\sigma_{11}\sigma_{22}}{\left(\sigma_{11}^{fT}\right)^2} < 1$$

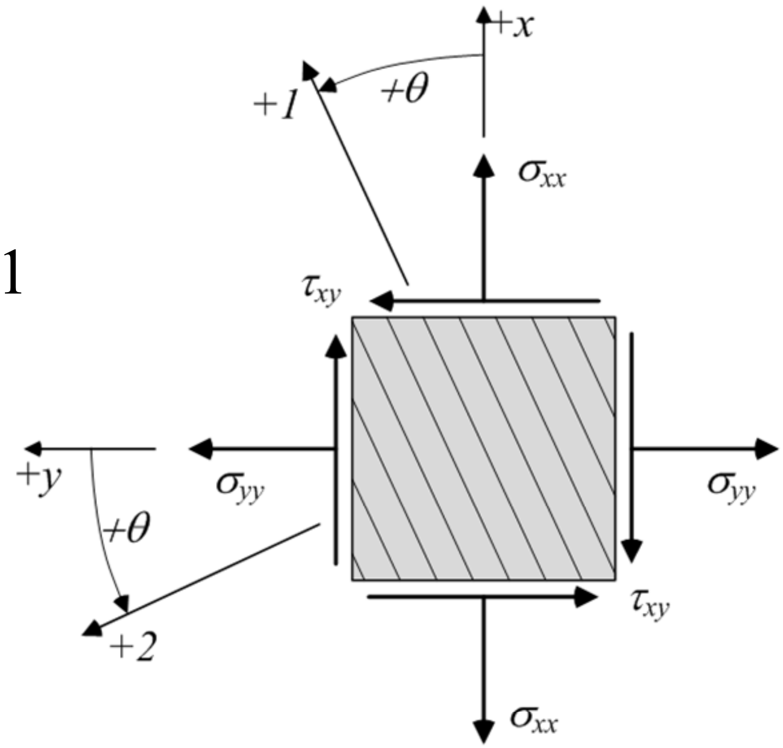


Tsai-Wu Failure Criterion

(Plane stress form)

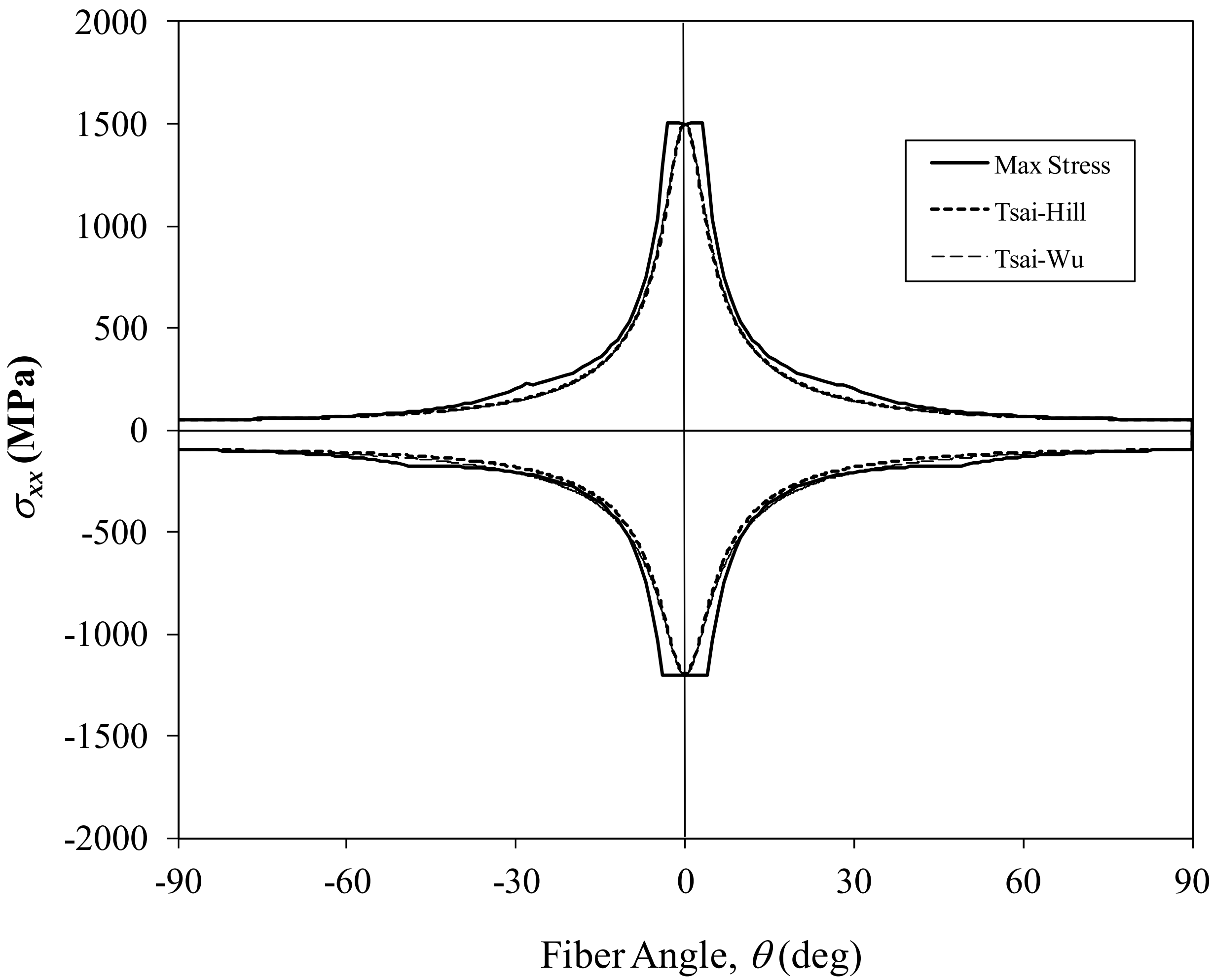
- Failure does not occur if:

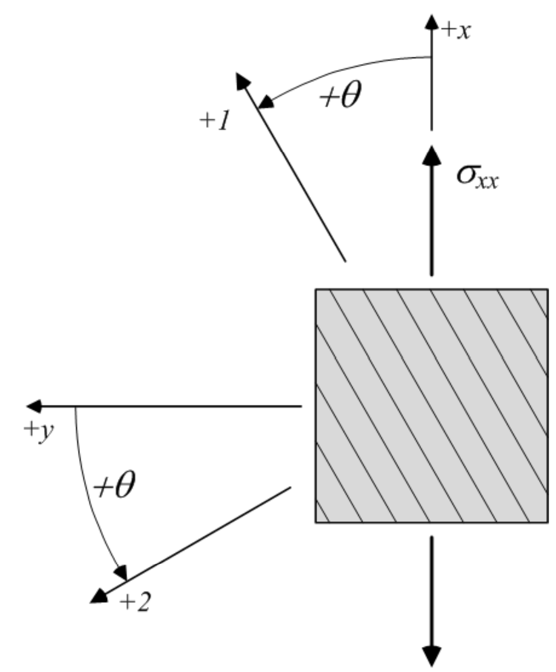
$$X_1\sigma_{11} + X_2\sigma_{22} + X_{11}\sigma_{11}^2$$
$$+ X_{22}\sigma_{22}^2 + X_{66}\tau_{12}^2 + 2X_{12}\sigma_{11}\sigma_{22} < 1$$



Macroscopic Failure Theories: Comparisons

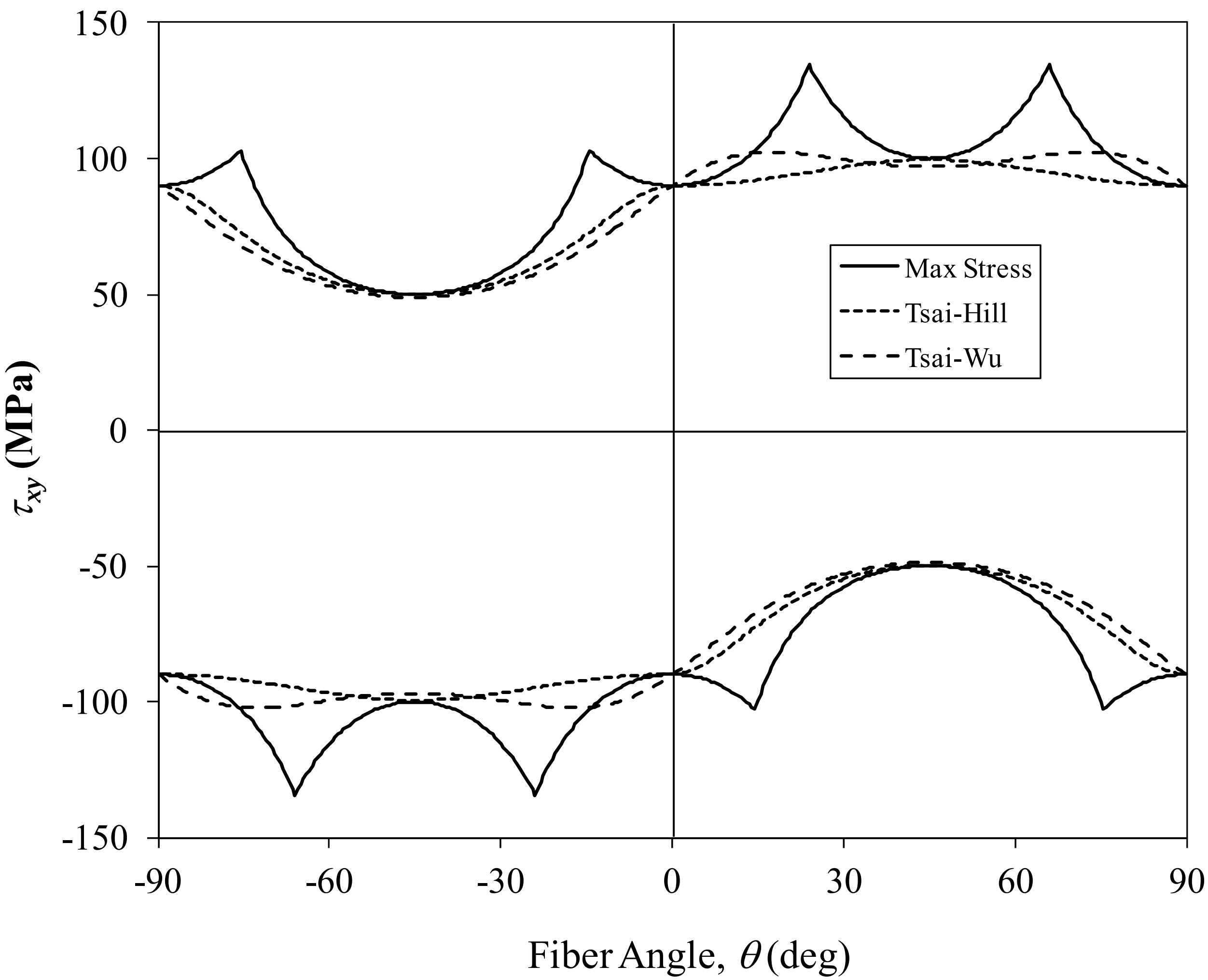
Off-axis Gr/Ep specimen subject to uniaxial stress

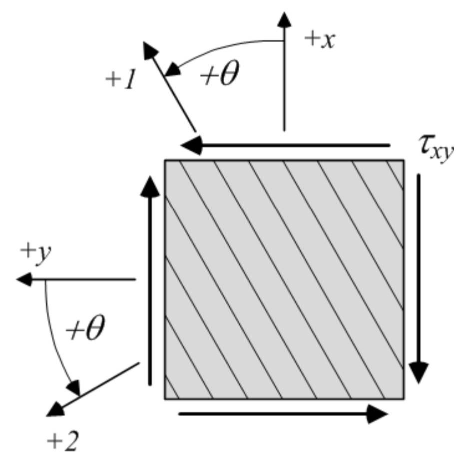




Macroscopic Failure Theories: Comparisons

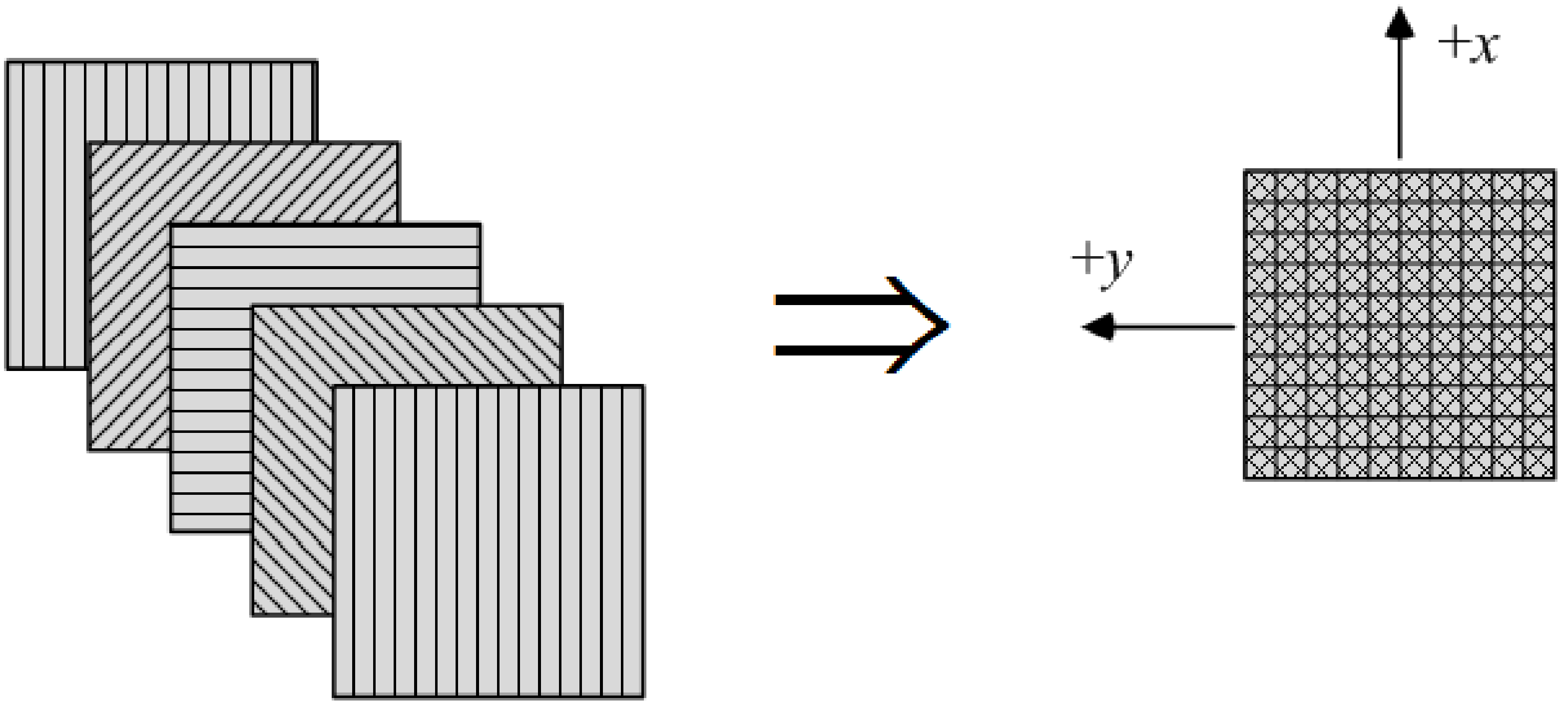
Off-axis Gr/Ep specimen subjected to pure shear stress



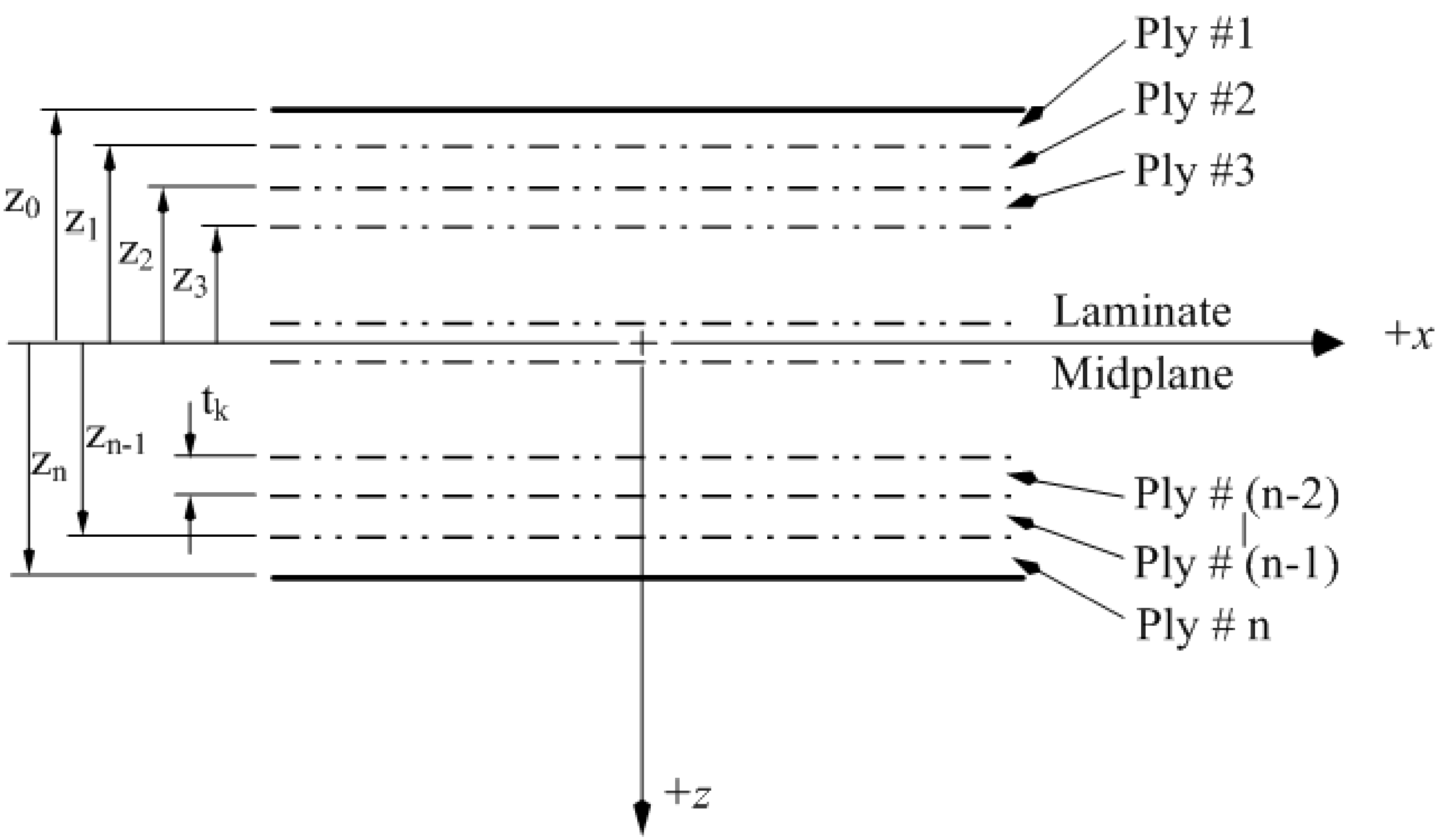


Chapter 6: Thermomechanical Behavior of Multiangle Composite Laminates

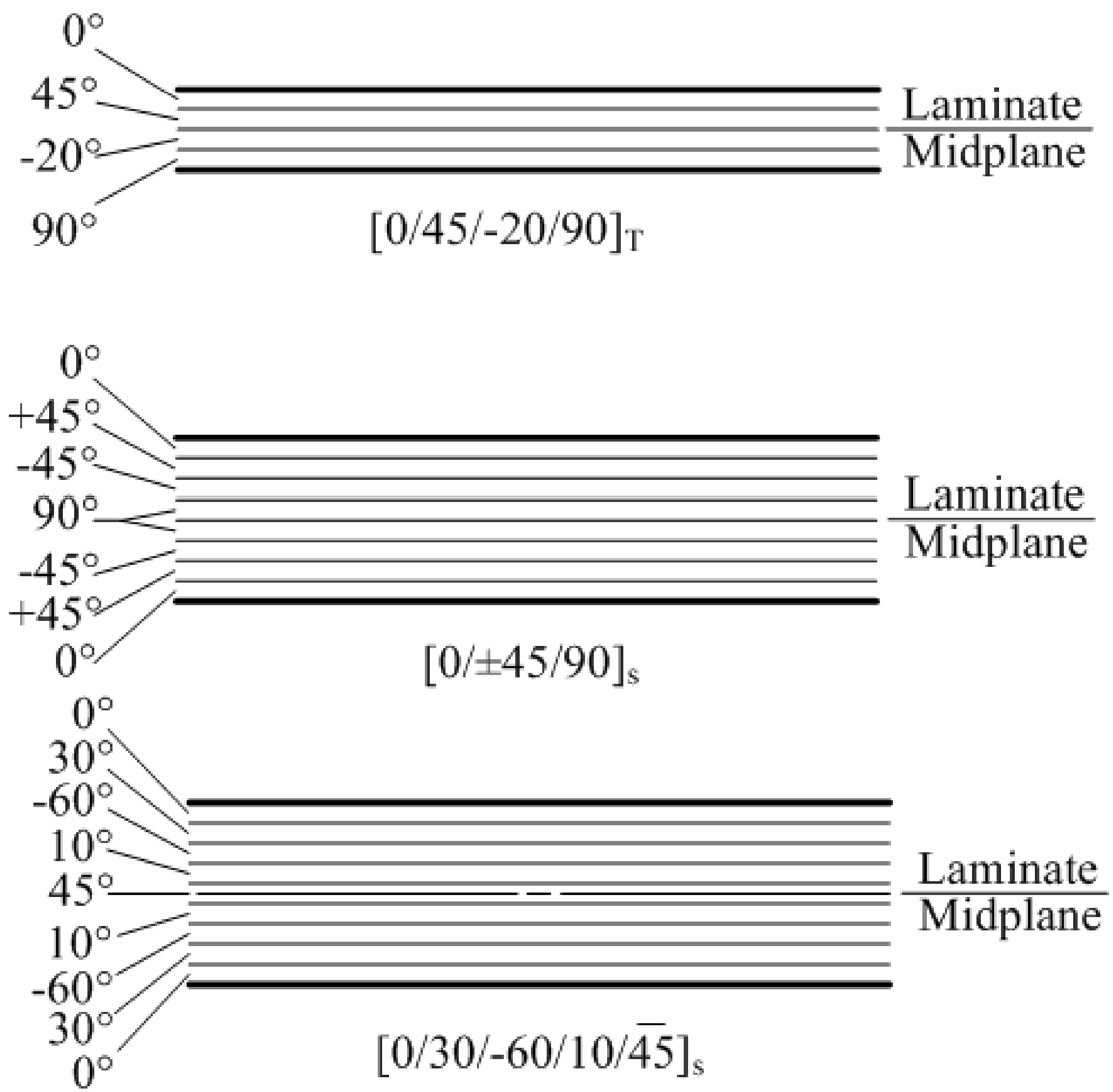
(CLT = Classical Lamination Theory)



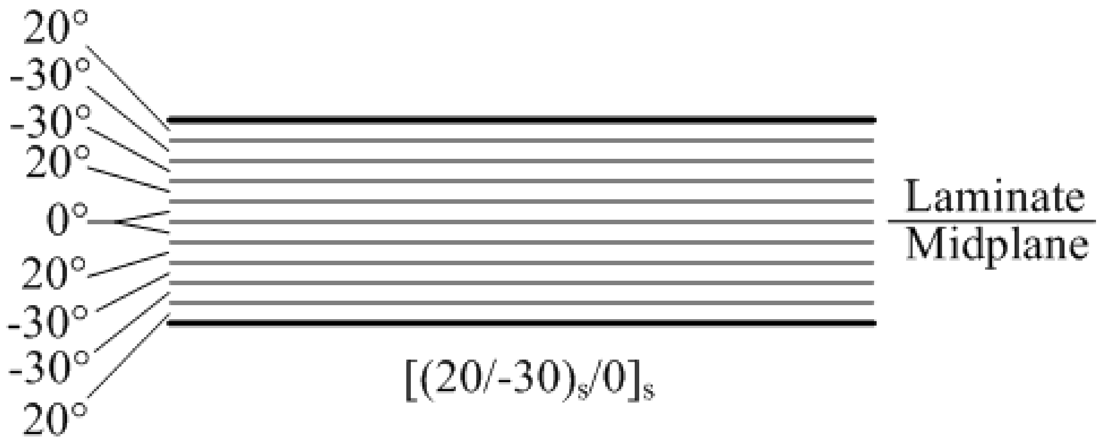
Defining Ply Interface Positions

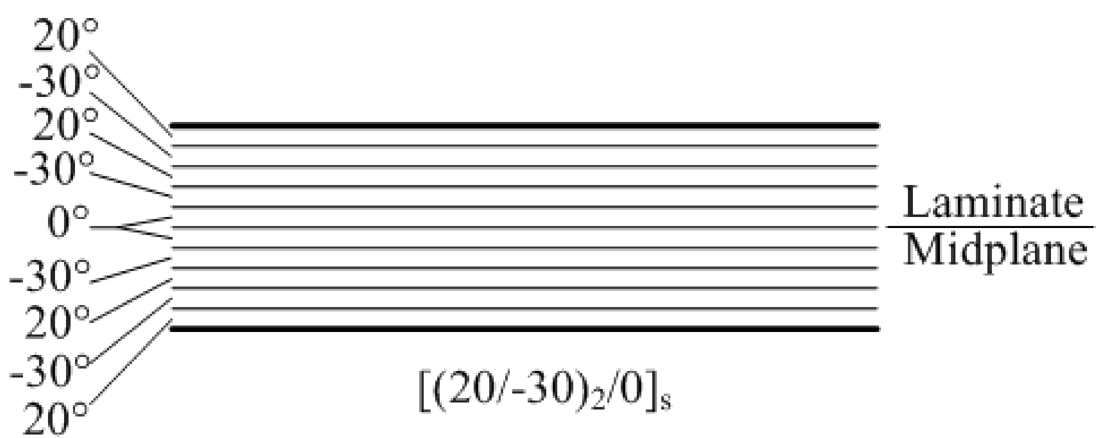


Describing Stacking Sequences



Describing Stacking Sequences





Kirchhoff Hypothesis

"a straight line which is initially perpendicular to the midplane of a thin plate remains straight and perpendicular to the midplane after deformation"

- Ultimately allows us to calculate the strain at any through-thickness position z:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

where:

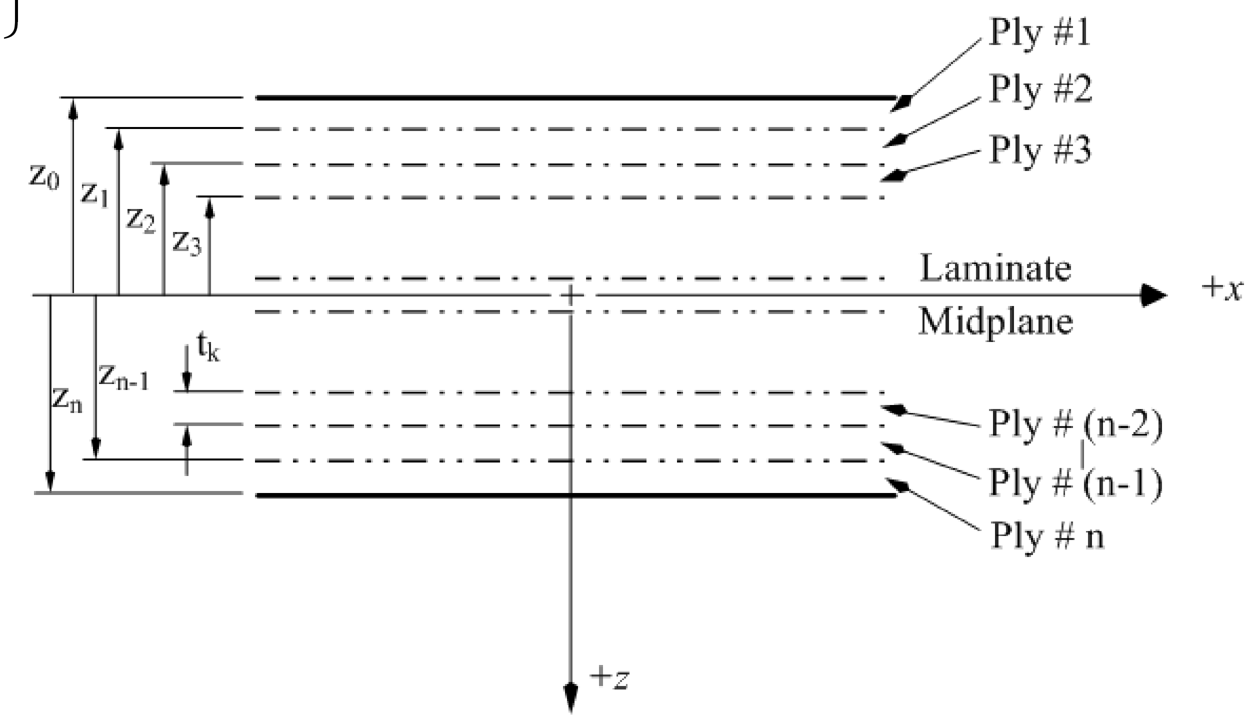
$$\varepsilon_{xx}^{o}, \varepsilon_{yy}^{o}, \gamma_{xy}^{o} = \text{midplane strains}$$

$$\kappa_{xx}, \kappa_{yy}, \kappa_{xy} = \text{midplane curvatures}$$

Ply Strains

- Strains at ply interfaces are usually of greatest interest:

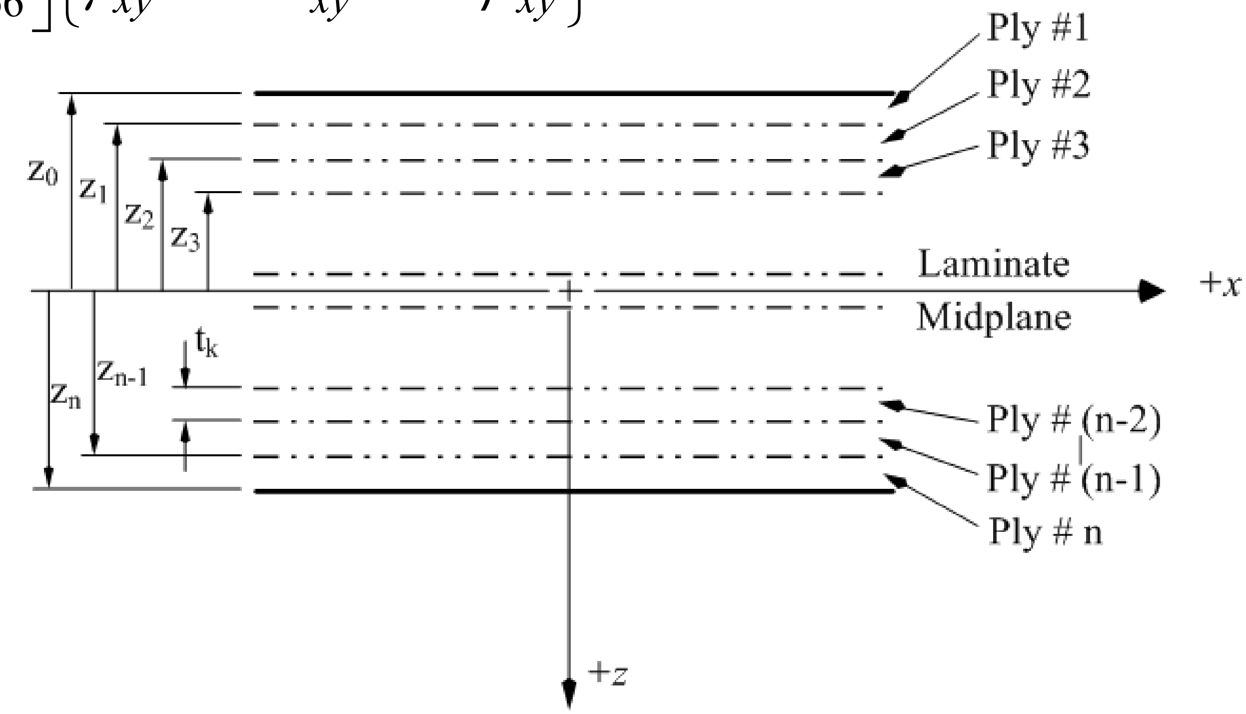
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$



Ply Stresses

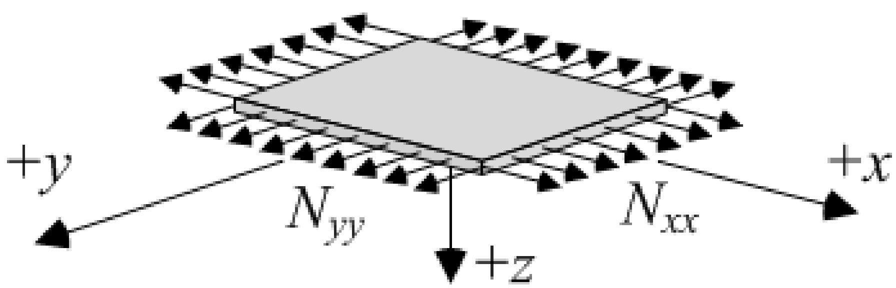
- Ply stresses can be calculated using Hooke's law:

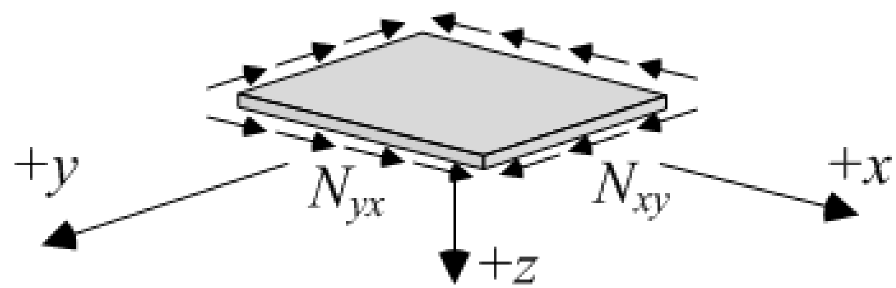
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy} \end{Bmatrix}$$



Laminate Loading

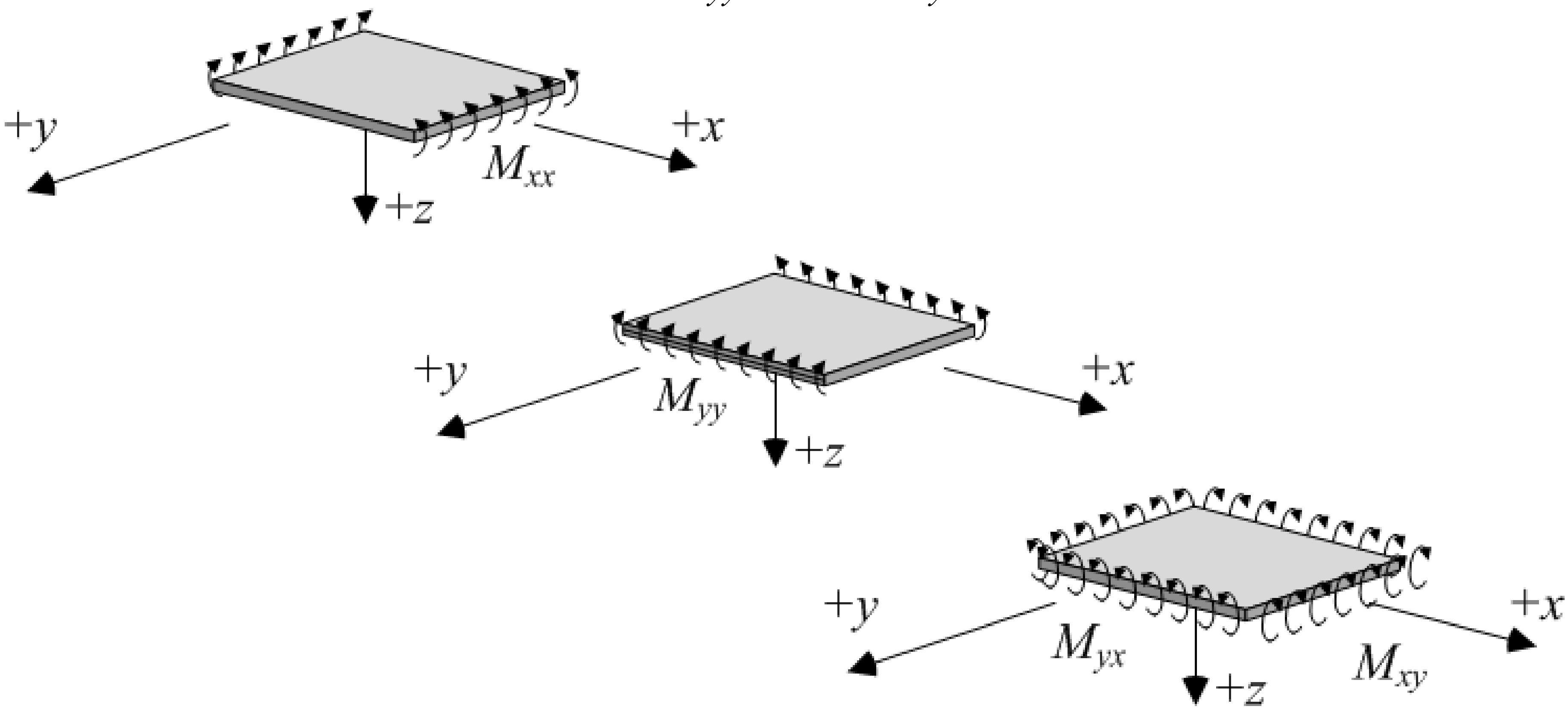
- Loads considered are restricted to those that lead to plane stress conditions…two types:
  - Stress resultants, N_{xx}, N_{yy} , and N_{xy}….units = force/length





Laminate Loading

- Loads considered are restricted to those that lead to plane stress conditions…two types:
  - Moment resultants, M_{xx}, M_{yy} , and M_{xy}….units = force-length/length



Laminate Loading

- It can be shown:

$$N_{xx} = \int_{-t/2}^{t/2} \sigma_{xx}\, dz$$

- Hooke's law:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy} \end{Bmatrix}$$

- Substituting for σ_{xx} and integrating in piece-wise fashion:

$$N_{xx} = A_{11}\varepsilon_{xx}^{o} + A_{12}\varepsilon_{yy}^{o} + A_{16}\gamma_{xy}^{o} + B_{11}\kappa_{xx} + B_{12}\kappa_{yy} + B_{16}\kappa_{xy} - N_{xx}^{T} - N_{xx}^{M}$$

Laminate Loading

- Where:

$$A_{11} = \left\{\left(\overline{Q}_{11}\right)_1 [z_1 - z_0] + \left(\overline{Q}_{11}\right)_2 [z_2 - z_1] + \left(\overline{Q}_{11}\right)_3 [z_3 - z_2] + ... + \left(\overline{Q}_{11}\right)_n [z_n - z_{n-1}]\right\}$$

$$A_{12} = \left\{\left(\overline{Q}_{12}\right)_1 [z_1 - z_0] + \left(\overline{Q}_{12}\right)_2 [z_2 - z_1] + \left(\overline{Q}_{12}\right)_3 [z_3 - z_2] + ... + \left(\overline{Q}_{12}\right)_n [z_n - z_{n-1}]\right\}$$

$$A_{16} = \left\{\left(\overline{Q}_{16}\right)_1 [z_1 - z_0] + \left(\overline{Q}_{16}\right)_2 [z_2 - z_1] + \left(\overline{Q}_{16}\right)_3 [z_3 - z_2] + ... + \left(\overline{Q}_{16}\right)_n [z_n - z_{n-1}]\right\}$$

$$B_{11} = \frac{1}{2}\left\{\left(\overline{Q}_{11}\right)_1 [z_1^2 - z_0^2] + \left(\overline{Q}_{11}\right)_2 [z_2^2 - z_1^2] + \left(\overline{Q}_{11}\right)_3 [z_3^2 - z_2^2] + ... + \left(\overline{Q}_{11}\right)_n [z_n^2 - z_{n-1}^2]\right\}$$

$$B_{12} = \frac{1}{2}\left\{\left(\overline{Q}_{12}\right)_1 [z_1^2 - z_0^2] + \left(\overline{Q}_{12}\right)_2 [z_2^2 - z_1^2] + \left(\overline{Q}_{12}\right)_3 [z_3^2 - z_2^2] + ... + \left(\overline{Q}_{12}\right)_n [z_n^2 - z_{n-1}^2]\right\}$$

$$B_{16} = \frac{1}{2}\left\{\left(\overline{Q}_{16}\right)_1 [z_1^2 - z_0^2] + \left(\overline{Q}_{16}\right)_2 [z_2^2 - z_1^2] + \left(\overline{Q}_{16}\right)_3 [z_3^2 - z_2^2] + ... + \left(\overline{Q}_{16}\right)_n [z_n^2 - z_{n-1}^2]\right\}$$

$$N_{xx}^T \equiv \Delta T \sum_{k=1}^{n} \left\{\left[\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy}\right]_k [z_k - z_{k-1}]\right\}$$

$$N_{xx}^M \equiv \Delta M \sum_{k=1}^{n} \left\{\left[\overline{Q}_{11}\beta_{xx} + \overline{Q}_{12}\beta_{yy} + \overline{Q}_{16}\beta_{xy}\right]_k [z_k - z_{k-1}]\right\}$$

Laminate Loading

- It can be shown:

$$M_{xx} = \int_{-t/2}^{t/2} \sigma_{xx}\, z dz$$

- Hooke's law:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy} \end{Bmatrix}$$

- Substituting for σ_{xx} and integrating in piece-wise fashion:

$$M_{xx} = B_{11}\varepsilon^{o}_{xx} + B_{12}\varepsilon^{o}_{yy} + B_{16}\gamma^{o}_{xy} + D_{11}\kappa_{xx} + D_{12}\kappa_{yy} + D_{16}\kappa_{xy} - M^{T}_{xx} - M^{M}_{xx}$$

Laminate Loading

- Where:

$$B_{11} = \frac{1}{2}\left\{\left(\overline{Q}_{11}\right)_1[z_1^2 - z_0^2] + \left(\overline{Q}_{11}\right)_2[z_2^2 - z_1^2] + \left(\overline{Q}_{11}\right)_3[z_3^2 - z_2^2] + ... + \left(\overline{Q}_{11}\right)_n[z_n^2 - z_{n-1}^2]\right\}$$

$$B_{12} = \frac{1}{2}\left\{\left(\overline{Q}_{12}\right)_1[z_1^2 - z_0^2] + \left(\overline{Q}_{12}\right)_2[z_2^2 - z_1^2] + \left(\overline{Q}_{12}\right)_3[z_3^2 - z_2^2] + ... + \left(\overline{Q}_{12}\right)_n[z_n^2 - z_{n-1}^2]\right\}$$

$$B_{16} = \frac{1}{2}\left\{\left(\overline{Q}_{16}\right)_1[z_1^2 - z_0^2] + \left(\overline{Q}_{16}\right)_2[z_2^2 - z_1^2] + \left(\overline{Q}_{16}\right)_3[z_3^2 - z_2^2] + ... + \left(\overline{Q}_{16}\right)_n[z_n^2 - z_{n-1}^2]\right\}$$

$$D_{11} = \frac{1}{3}\left\{\left(\overline{Q}_{11}\right)_1[z_1^3 - z_0^3] + \left(\overline{Q}_{11}\right)_2[z_2^3 - z_1^3] + \left(\overline{Q}_{11}\right)_3[z_3^3 - z_2^3] + ... + \left(\overline{Q}_{11}\right)_n[z_n^3 - z_{n-1}^3]\right\}$$

$$D_{12} = \frac{1}{3}\left\{\left(\overline{Q}_{12}\right)_1[z_1^3 - z_0^3] + \left(\overline{Q}_{12}\right)_2[z_2^3 - z_1^3] + \left(\overline{Q}_{12}\right)_3[z_3^3 - z_2^3] + ... + \left(\overline{Q}_{12}\right)_n[z_n^3 - z_{n-1}^3]\right\}$$

$$D_{16} = \frac{1}{3}\left\{\left(\overline{Q}_{16}\right)_1[z_1^3 - z_0^3] + \left(\overline{Q}_{16}\right)_2[z_2^3 - z_1^3] + \left(\overline{Q}_{16}\right)_3[z_3^3 - z_2^3] + ... + \left(\overline{Q}_{16}\right)_n[z_n^3 - z_{n-1}^3]\right\}$$

$$M_{xx}^{T} \equiv \frac{\Delta T}{2}\sum_{k=1}^{n}\left\{\left[\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy}\right]_k\left[z_k^2 - z_{k-1}^2\right]\right\}$$

$$M_{xx}^{M} \equiv \frac{\Delta M}{2}\sum_{k=1}^{n}\left\{\left[\overline{Q}_{11}\beta_{xx} + \overline{Q}_{12}\beta_{yy} + \overline{Q}_{16}\beta_{xy}\right]_k\left[z_k^2 - z_{k-1}^2\right]\right\}$$

Laminate Loading

- Process repeated for all stress and moment resultants, finally resulting in:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{T} \\ N_{yy}^{T} \\ N_{xy}^{T} \\ M_{xx}^{T} \\ M_{yy}^{T} \\ M_{xy}^{T} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{M} \\ N_{yy}^{M} \\ N_{xy}^{M} \\ M_{xx}^{M} \\ M_{yy}^{M} \\ M_{xy}^{M} \end{Bmatrix}$$

Laminate Loading

- Inverting:

$$\begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{21} & b_{22} & b_{26} \\ a_{16} & a_{26} & a_{66} & b_{61} & b_{62} & b_{66} \\ b_{11} & b_{21} & b_{61} & d_{11} & d_{12} & d_{16} \\ b_{12} & b_{22} & b_{62} & d_{12} & d_{22} & d_{26} \\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} N_{xx} + N_{xx}^{T} + N_{xx}^{M} \\ N_{yy} + N_{yy}^{T} + N_{yy}^{M} \\ N_{xy} + N_{xy}^{T} + N_{xy}^{M} \\ M_{xx} + M_{xx}^{T} + M_{xx}^{M} \\ M_{yy} + M_{yy}^{T} + M_{yy}^{M} \\ M_{xy} + M_{xy}^{T} + M_{xy}^{M} \end{Bmatrix}$$

Simplifications Due to Stacking Sequence

- Various terms within the $[ABD]$ and $[abd]$ matrices are always zero for certain stacking sequences (see Section 6.7)
- The most important simplification occurs for *symmetric* laminates…in this case:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{T} \\ N_{yy}^{T} \\ N_{xy}^{T} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{M} \\ N_{yy}^{M} \\ N_{xy}^{M} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifications Due to Stacking Sequence

- Various terms within the $[ABD]$ and $[abd]$ matrices are always zero for certain stacking sequences (see Section 6.7)
- The most important simplification occurs for *symmetric* laminates…in this case:

$$\begin{Bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{yy} \\ \gamma^o_{xy} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{26} & 0 & 0 & 0 \\ a_{16} & a_{26} & a_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{11} & d_{12} & d_{16} \\ 0 & 0 & 0 & d_{12} & d_{22} & d_{26} \\ 0 & 0 & 0 & d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} N_{xx} + N^T_{xx} + N^M_{xx} \\ N_{yy} + N^T_{yy} + N^M_{yy} \\ N_{xy} + N^T_{xy} + N^M_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}$$

Effective Laminate Properties

- Effective elastic properties of a laminate can be determined by applying the “normal” definitions:

- Extensional (in-plane):

$$\bar{E}_{xx}^{ex} = \frac{1}{ta_{11}} \qquad \bar{\nu}_{xy}^{ex} = \frac{-a_{12}}{a_{11}} \qquad \bar{\eta}_{xx,xy}^{ex} = \frac{a_{16}}{a_{11}}$$

$$\bar{E}_{yy}^{ex} = \frac{1}{ta_{22}} \qquad \bar{\nu}_{yx}^{ex} = \frac{-a_{12}}{a_{22}} \qquad \bar{\eta}_{yy,xy}^{ex} = \frac{a_{26}}{a_{22}}$$

$$\bar{G}_{xy} = \frac{1}{ta_{66}} \qquad \bar{\eta}_{xy,xx} = \frac{a_{16}}{a_{66}} \qquad \bar{\eta}_{xy,yy} = \frac{a_{26}}{a_{66}}$$

Effective Laminate Properties

- Effective elastic properties of a laminate can be determined by applying the "normal" definitions

- Flexural (bending):

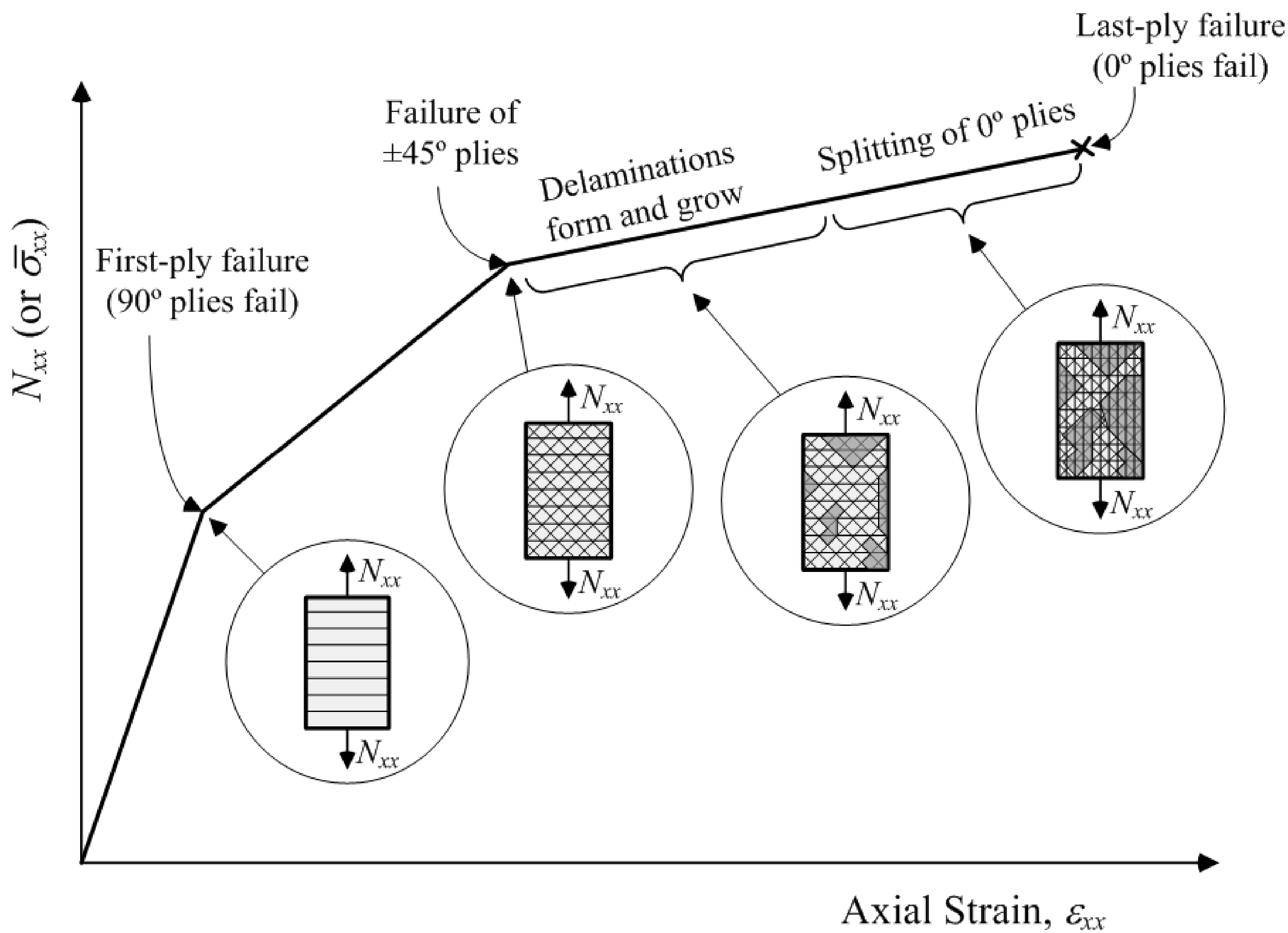
$$\overline{E}_{xx}^{fl} = \frac{12}{t^3 d_{11}} \qquad \overline{\nu}_{xy}^{fl} = \frac{-d_{12}}{d_{11}} \qquad \overline{\eta}_{xx,xy}^{fl} = \frac{d_{16}}{d_{11}}$$

$$\overline{E}_{yy}^{fl} = \frac{12}{t^3 d_{22}} \qquad \overline{\nu}_{yx}^{fl} = \frac{-d_{12}}{d_{22}} \qquad \overline{\eta}_{yy,xy}^{fl} = \frac{d_{26}}{d_{22}}$$

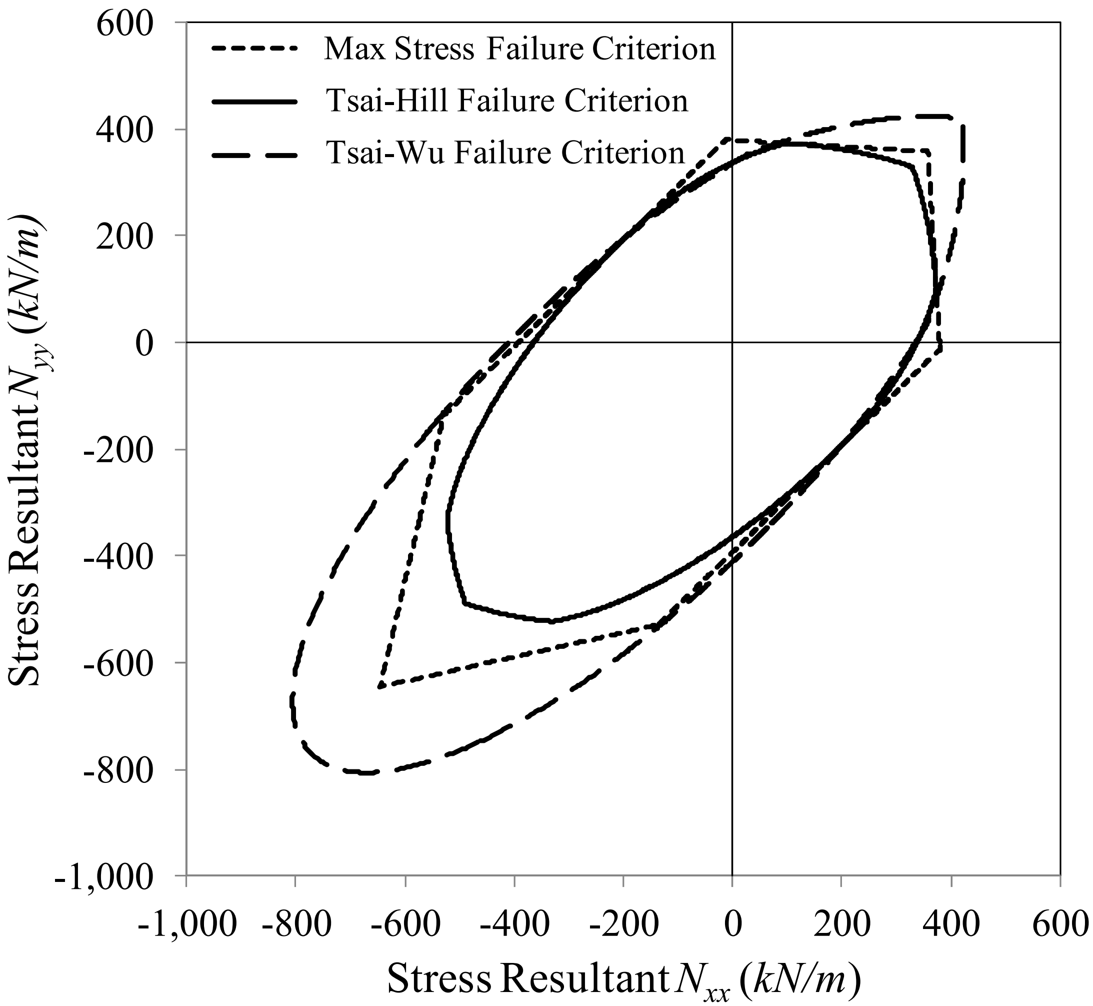
Program CLT

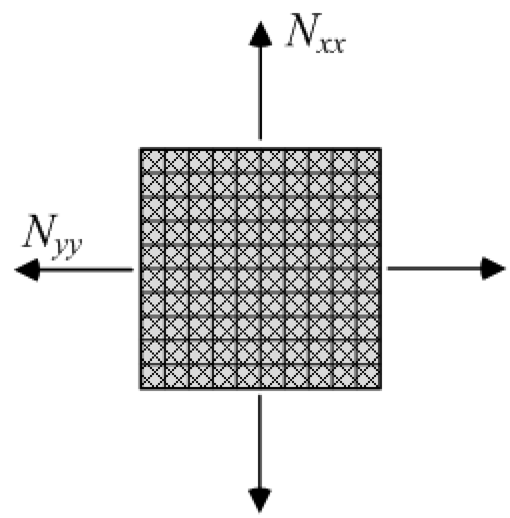
- Most of the topics included in this review are implemented in the program CLT (*C*lassical *L*amination *T*heory)

Chapter 7: Failure of Multiangle Laminates

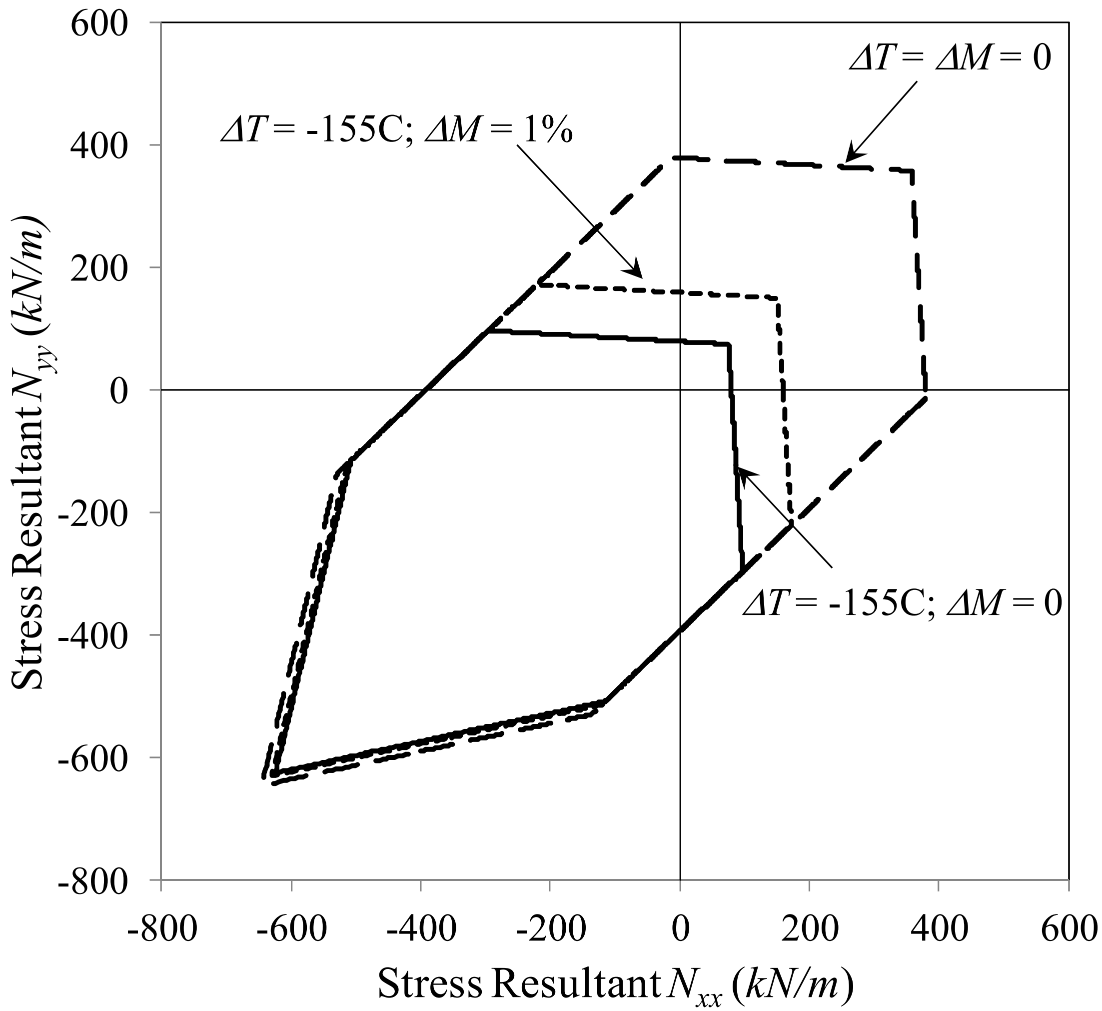


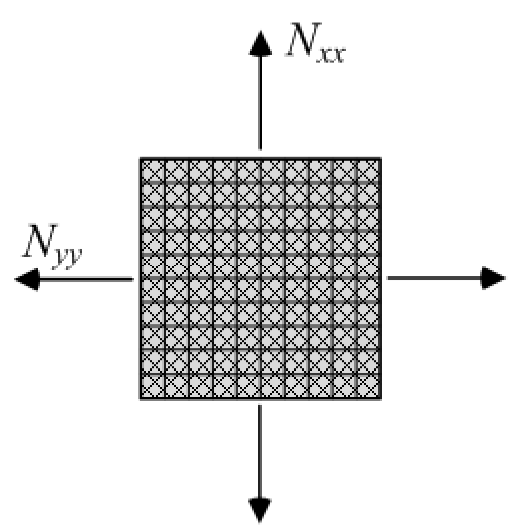
First-ply Failure Loads and/or First-ply Failure Envelopes can be Predicted Using Program LAMFAIL



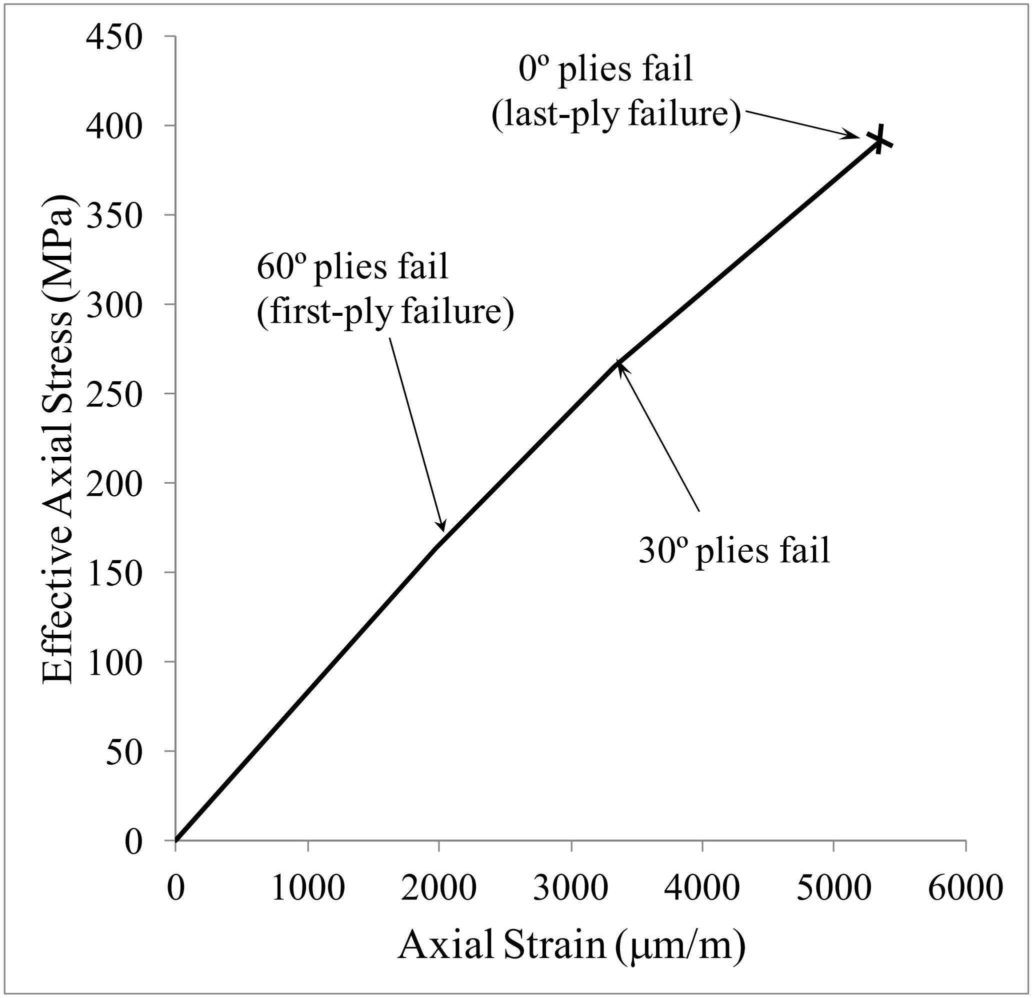


Environmental Effects Dramatically Affect Predicted First-ply Failure Loads





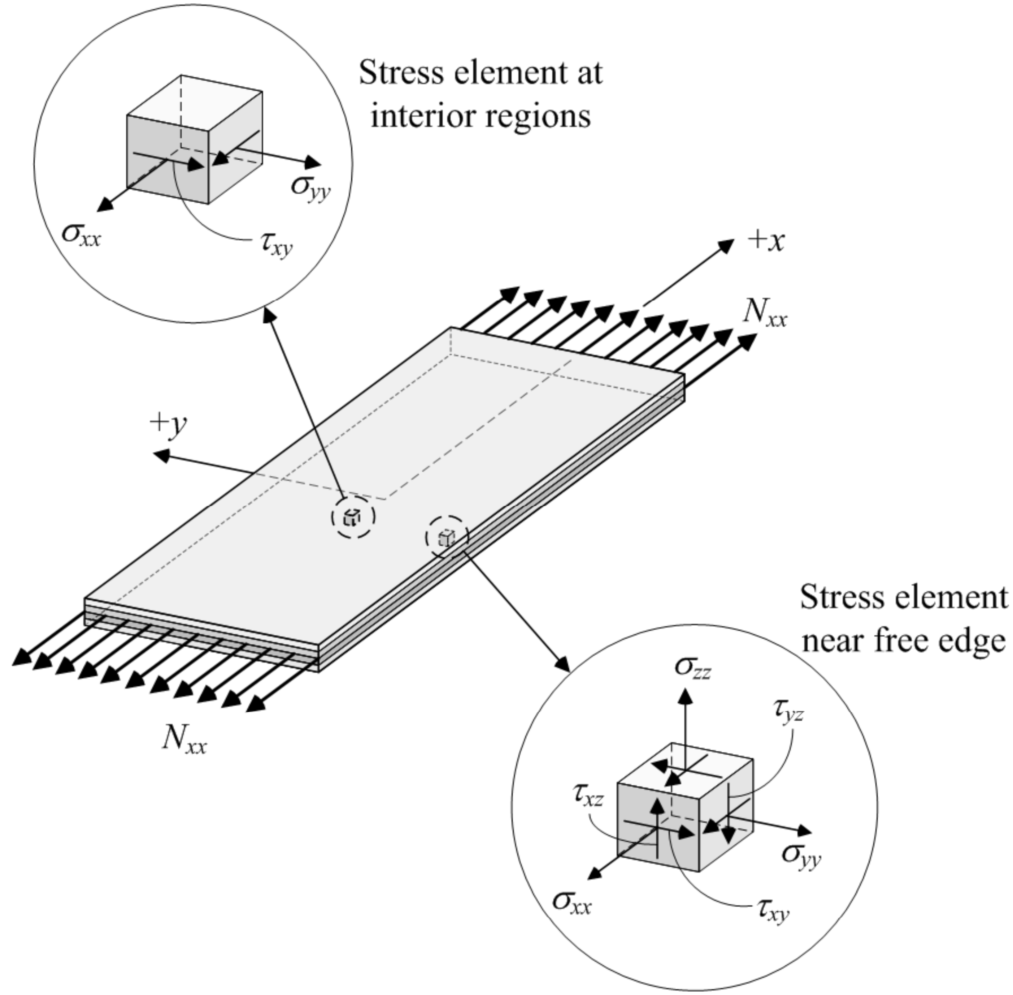
Last-Ply Failure Loads Can be Predicted Using the Ply-Discount Scheme (Program LAMFAIL)



Predicted stress-strain curve for a $[0/30/60]_s$ laminate

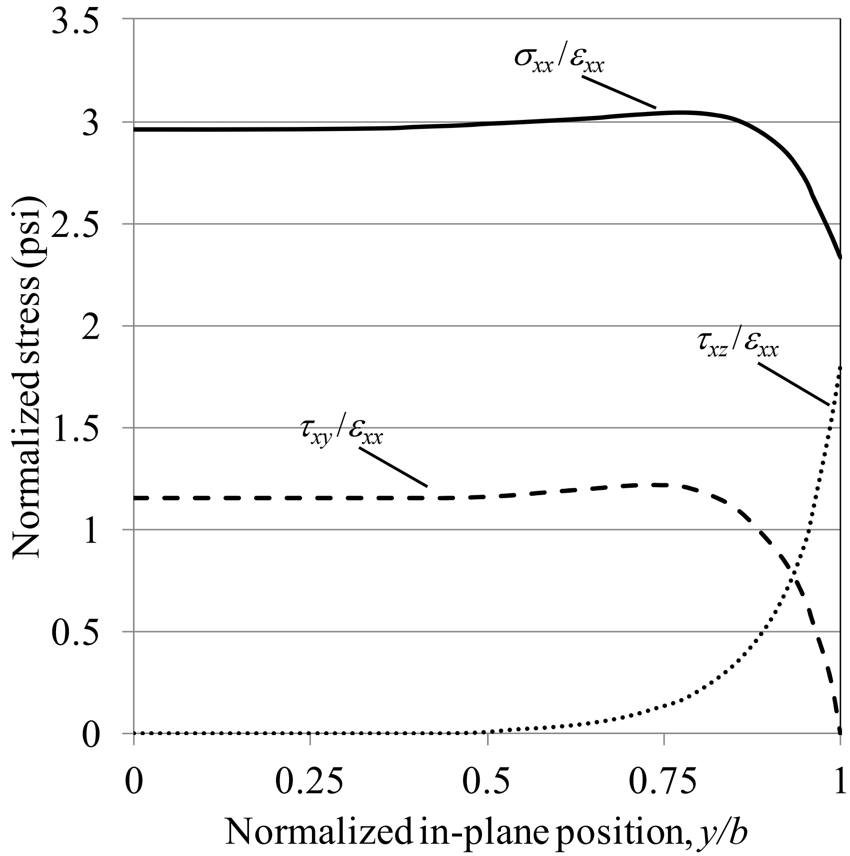
3-D Stress-State Exists Near a Free-edge and Complicate Failure Predictions

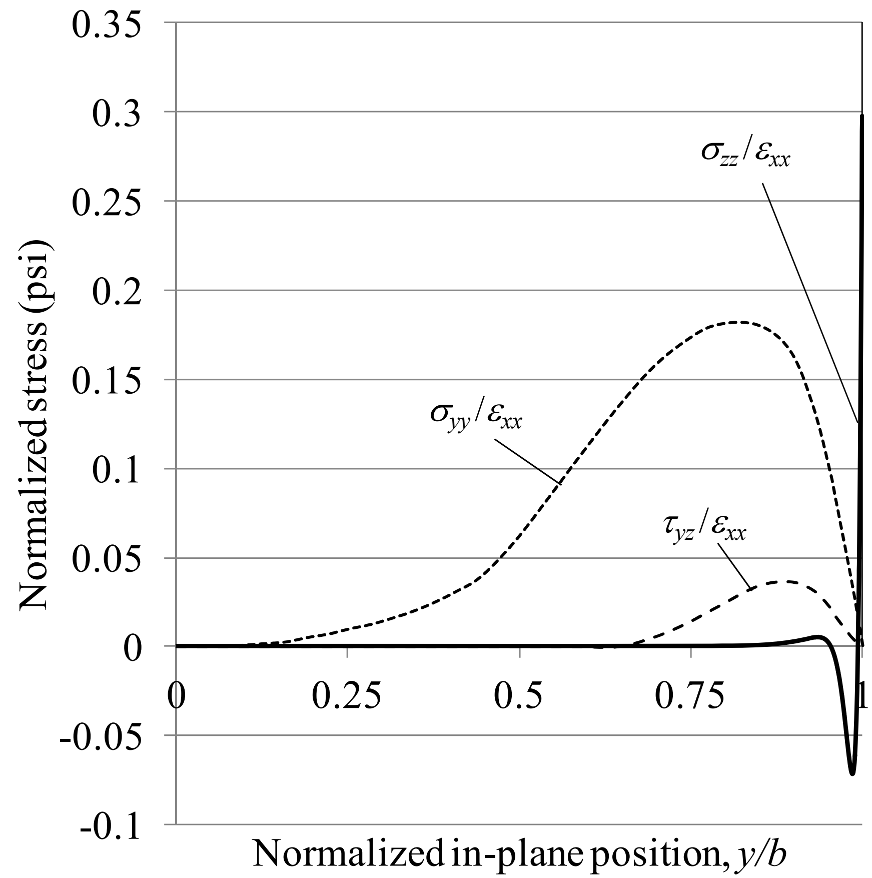
(see Section 6.13)



3-D Stress-State Exists Near a Free-edge and Complicate Failure Predictions *(see Section 6.13)*

(Typical results for a $[45/-45]_s$ laminate subject to uniaxial loading)





Good luck on all your finals,
and have a great spring break!