ME 477 Embedded Computing

Nyquist-Shannon Sampling Theorem

Suppose that we wish to sample, with uniform period T, a continuous-time, band-limited signal. What should the sampling frequency $f_s = 1/T$ be such that the sequence of samples faithfully represents the continuous signal?

Begin by describing the sampling processes mathematically as shown in Fig. 1. Represent sampling in the time domain as the multiplication of the continuous signal by an infinitely long train of equally spaced impulses.

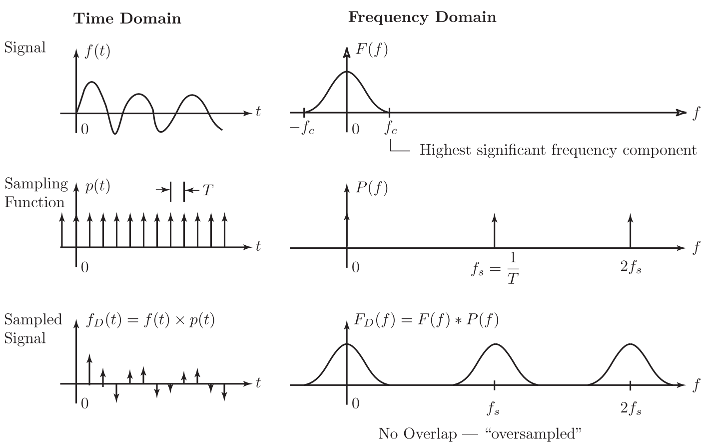

Figure 1: Time and Frequency domain representations of sampling. (Oversampled)

Recall that the Fourier transform of the product of two time functions is the convolution of their corresponding transforms:

$$
\begin{aligned}
F_d(f) = \mathscr{F}[f(t) \times p(t)] &= F(f) * P(f) \\
&= \int_{-\infty}^{+\infty} F(f-x)P(x)dx
\end{aligned}
$$

where $\mathscr{F}[\ ]$ is the Fourier transform operator, $F(f) = \mathscr{F}[F(t)]$, $P(f) = \mathscr{F}[p(t)]$, and $*$ denotes convolution. As shown in the frequency domain, we assume that the continuous-time signal is band-limited with no frequency components higher than f_c.

Note that the Fourier transform of the sampling function $p(t)$ is an infinite train of equally spaced impulses with spacing $f_s = 1/T$. The convolution yields the transform of the sampled signal as the superposition of the transform of the continuous signal repeated with spacing $f_s = 1/T$. Or generally, the **transform of a sampled signal**:

Theorem. *The Fourier transform of a sampled signal is equal to the superposition of the transform of the continuous signal shifted by all multiples of the sampling frequency.*

Examining the transform of the sampled signal $F_d(f)$, we see that the copy of $F(f)$ near $f = 0$ is identical to that of the original signal $f(t)$. Therefore, we could inverse transform the copy near $f = 0$ to reconstruct the entire continuous-time function $f(t)$ with perfect fidelity!

Now assume that, in comparison with Fig. 1, we decrease the sampling frequency (increase the sampling period). See Fig. 2. The increased spacing of the sampling function in the time domain corresponds to decreased spacing in the frequency domain. In this figure, the copies of $F(f)$ overlap, and superposition of the copies of $F(f)$ (solid curve) no longer resembles $F(f)$ near $f = 0$.

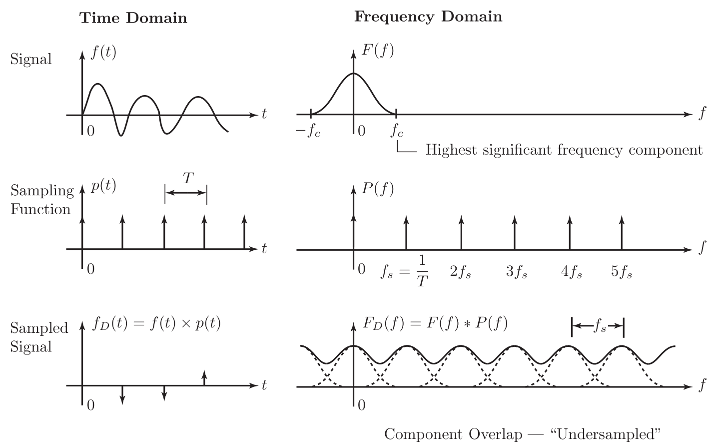

Figure 2: Low frequency sampling. (Undersampled)

Clearly, to avoid the overlap, the sampling frequency f_s must be greater than $2 \times f_c$. This leads to a *sufficient* condition for sampling band-limited signals, the **Nyquist-Shannon sampling theorem**:

Theorem. *If a continuous-time signal $f(t)$ contains no frequency components higher than f_c, it is completely determined by sampling at a frequency f_s greater than $2 \times f_c$.*

The threshold $f_s/2$ is called the **Nyquist frequency**. For proper sampling, all frequency components of $f(t)$ must be below the Nyquist frequency.