Homework #2, ME/MSE 485, due on Jan. 27, 2011

1. Consider the thermal composite composed of two kinds of conductors, K_{f1}=100W/(Km) and K_{f2}=50W/(Km), and one insulator, K_m=0.2W/(Km), see the following figure. Under applied heat at the top, Q=100W, we would like to calculate the temperature at top, T_1, and the temperature at mid-points, T_2-T_4, where the temperature at the bottom, T_5 is set to T_5=0. Answer the following questions. Please note that you do not need to solve for all unknowns.
   (a) Calculate all thermal resistances, R_{ij} defined in the figure, assuming the thickness perpendicular to this paper sheet is 1mm.

$$R_{12} = \frac{L}{K_{f1} \times A} = \frac{0.5 \times 10^{-3}}{100 \times 1 \times 10^{-6}} = 5(K/W) = R_{25}$$

$$R_{13} = \frac{L}{K_m \times A} = \frac{0.5 \times 10^{-3}}{0.2 \times 1 \times 10^{-6}} = 2.5 \times 10^3 (K/W) = R_{35}$$

$$R_{14} = \frac{L}{K_{f2} \times A} = \frac{0.5 \times 10^{-3}}{50 \times 1 \times 10^{-6}} = 10(K/W) = R_{45}$$

$$R_{23} = \frac{L_{f1}}{K_{f1} \times A} + \frac{L_m}{K_m \times A} = \frac{0.5 \times 10^{-3}}{100 \times 1 \times 10^{-6}} + \frac{0.5 \times 10^{-3}}{0.2 \times 1 \times 10^{-6}} = 2.505 \times 10^3 (K/W)$$

$$R_{34} = \frac{L_{f2}}{K_{f2} \times A} + \frac{L_m}{K_m \times A} = \frac{0.5 \times 10^{-3}}{50 \times 1 \times 10^{-6}} + \frac{0.5 \times 10^{-3}}{0.2 \times 1 \times 10^{-6}} = 2.51 \times 10^3 (K/W)$$

   (b) Set the algebraic equations at nodal points (1-5) by using Kirchoff Current Law: all currents (or thermal flow in this problem) coming to i-th nodal point if they are summed up, it is equal to zero.

For node 2,

$$\frac{T_1 - T_2}{R_{12}} = \frac{T_2 - T_5}{R_{25}} + \frac{T_2 - T_3}{R_{23}}$$

$$\Rightarrow \frac{T_1 - T_2}{5} = \frac{T_2 - 0}{5} + \frac{T_2 - T_3}{2.505 \times 10^3}$$

For node 3,

$$\frac{T_1 - T_3}{R_{13}} + \frac{T_2 - T_3}{R_{23}} + \frac{T_4 - T_3}{R_{34}} = \frac{T_3 - T_5}{R_{35}}$$

$$\Rightarrow \quad \frac{T_1 - T_3}{2.5\times 10^3} + \frac{T_2 - T_3}{2.505\times 10^3} + \frac{T_4 - T_3}{2.51\times 10^3} = \frac{T_3 - 0}{2.5\times 10^3}$$

For node 4,

$$\frac{T_1 - T_4}{R_{14}} = \frac{T_4 - T_5}{R_{45}} + \frac{T_4 - T_3}{R_{34}}$$

$$\Rightarrow \quad \frac{T_1 - T_4}{10} = \frac{T_4 - 0}{10} + \frac{T_4 - T_3}{2.51\times 10^3}$$

For node 1,

$$\frac{T_1 - T_2}{R_{12}} + \frac{T_1 - T_3}{R_{13}} + \frac{T_1 - T_4}{R_{14}} = Q$$

$$\Rightarrow \quad \frac{T_1 - T_2}{5} + \frac{T_1 - T_3}{2.5\times 10^3} + \frac{T_1 - T_4}{10} = 100(W)$$

(c) Solve for temperatures, T_1, T_2, T_3 and T_4

Because $\frac{1}{2500}$, $\frac{1}{2505}$ and $\frac{1}{2510}$ are relatively smaller than $\frac{1}{5}$ and $\frac{1}{10}$. So rewrite equation in section (b),

$$T_1 - T_2 = T_2$$
$$T_1 - T_3 + T_2 - T_3 + T_4 - T_3 = T_3$$
$$T_1 - T_4 = T_4$$
$$10(T_1 - T_2) + 5(T_1 - T_4) = 5000$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 1 & -4 & 1 \\ 1 & 0 & 0 & -2 \\ 15 & -10 & 0 & -5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 1 & -4 & 1 \\ 1 & 0 & 0 & -2 \\ 15 & -10 & 0 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{15} \\ -\frac{5}{6} & 0 & -\frac{1}{6} & \frac{1}{15} \\ -\frac{11}{24} & -\frac{1}{4} & -\frac{7}{24} & \frac{1}{15} \\ -\frac{1}{3} & 0 & -\frac{2}{3} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 666.67 \\ 333.33 \\ 333.33 \\ 333.33 \end{bmatrix}$$

So T1=666.67C and T2=T3=T4=333.33C