Transient on TL for ME485 1/25/11

Important Subjects in Time-Domain Responses

(1) Unit step function responses
(2) Delta-I noise
(3) Finite rise-time pulse and Laplace transform technique
(4) Pulse on lossy transmission lines and dispersion
(5) Forward and backward coupled noise (To be covered in the coupled line)
(6) Time-domain measurement techniques

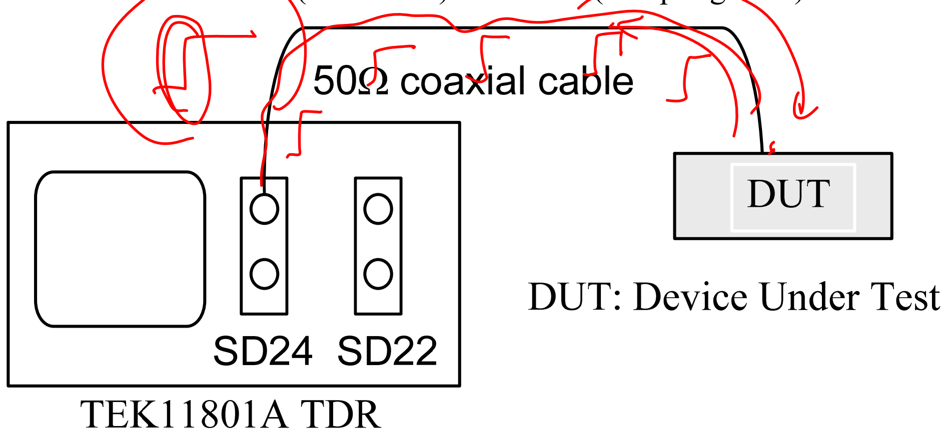
Ref: HP application note 1304-2 (Web page)

Time-domain Experiments

-Requires a very fast unit step function generator

-Requires a very fast oscilloscope

Lab Equipment:
TEK11801A TDR with SD24 (TDR head) and SD22 (Sampling head)



Frequency f

Experimental Setup for the Time Domain Analysis of Devices

$\lambda_0 = 530\,nm$.

Constant

$f \cdot \lambda = v$

Free space, $v = C_0$

~~$f \cdot \lambda_0 = C_0$~~

Water $v = \frac{C_0}{n}$

$n = \sqrt{\varepsilon_r}$ = index of refraction

$= 1.32$.

$u(t)$

$t.$

$\varepsilon_{effective} = 1.98$

equivalent circuit

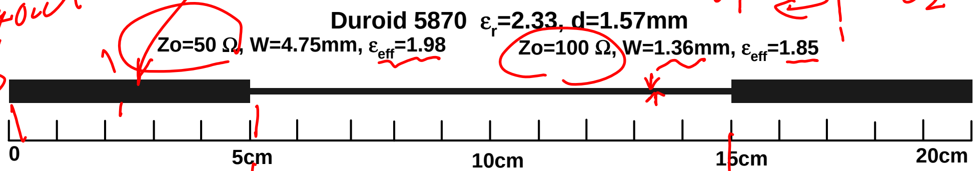
$Z_1=50$ $Z_2=100$ $Z_3=50$

v_1 v_2 v_3

Pictures of devices under test (DUT)

top layout

~~2 GHz~~



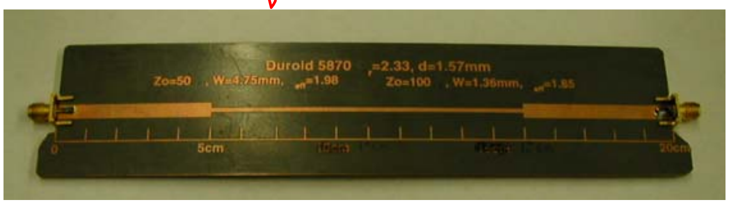
v = Velocity.

Free space c

$C_0 = 3\times10^8$ m/s

$v = \frac{C_0}{\sqrt{\varepsilon_r}}$

ε_r: dielectric constant.



Test PCB

Unknown device 1

$v = \frac{C_0}{\sqrt{\varepsilon_{effective}}}$

Unknown device 2

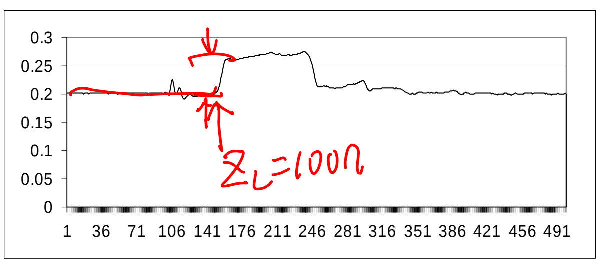
Unknown device 3

C

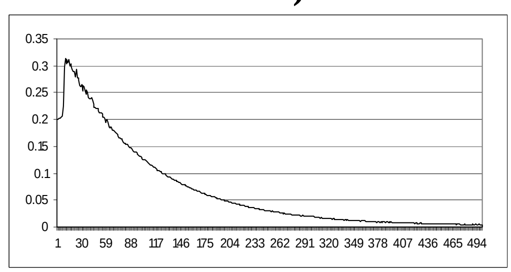
Expected responses (different TDR is used for these case)

L

R



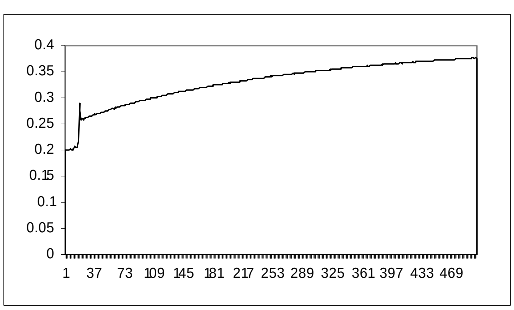
a) Unmatched Transmission Line



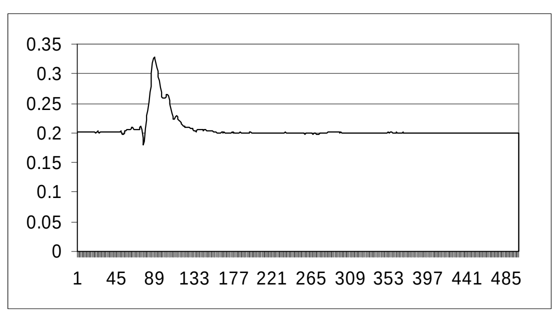
(b) Unknown1

(1 data point = 10ps)

L R



(c) Unknown 2



(d) Unknown 3

1. Unit Step Function Response from a Resistive Load

A digital signal on PCB (microstrip or stripline TL) is not a continuous wave. Rather it is a square (or pseudo square) pulse train which contains many frequency components. In this section, we will study the responses of matched and unmatched TL in time-domain. This is called "Transient Response".

Applications:

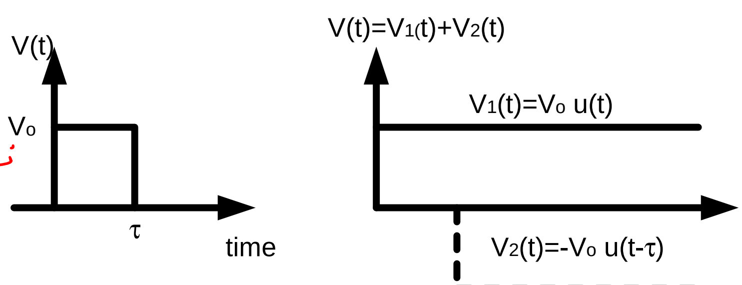
-TDR is an instrument which combines a high speed step function generator and fast oscilloscope scope. The reflection from an unmatched impedance can be detected. Example: Computer network.

-Optical TDR is similar to TDR but it uses a short pulse to find a faulty optical fiber cable.

Rectangular Pulse:

Assume the input signal is a pulse of duration τ which can be expressed as a sum of two unit step functions as shown below. Therefore, once we find the circuit response to a unit step function, we can obtain the circuit response to a rectangular pulse.

Reflection:



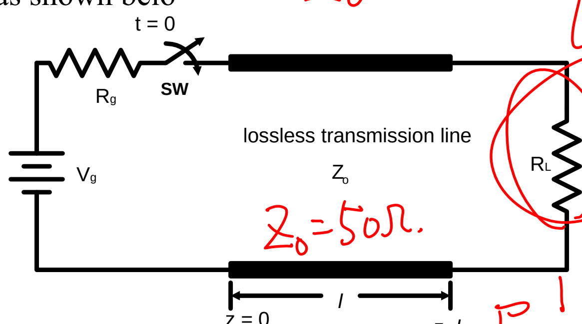
If $R = Z_0$, No reflection

The unit step function can be created by closing and opening a switch at the source side as shown belo

Z_0

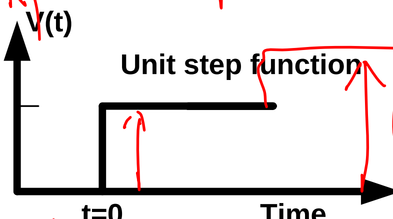
Load impedance.

$R = 100\Omega$.



$Z_L = R$

$Z_0 = 50\Omega$.



?

Γ_L

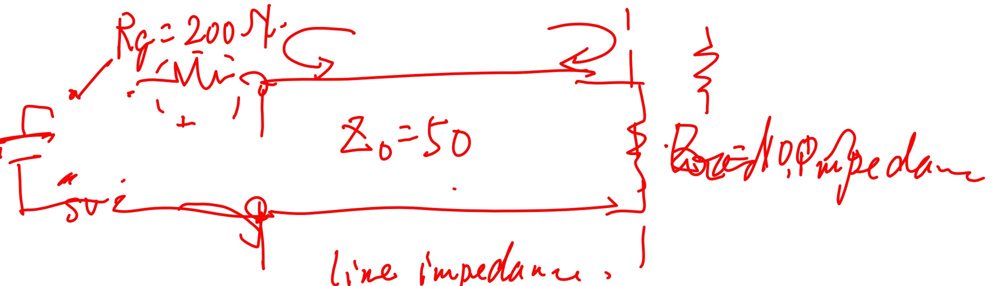
Amount of reflection.

\
Rg: source impedance
SW is closed at t=0

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0} = \frac{1}{3}$$

TL Response to a Unit Step Function

Rg: source impedance



SW : ON

Initial Condition

Let us assume the SW is closed at t=0. First, we need to find a initial voltage and current which propagate into the TL from the source. When the rise-time (voltage step) arrives at a TL, it does not see the load (R_L). Instead, it sees the TL which has a characteristic impedance of Z_o. Therefore, the voltage going into the TL is given by a voltage divider between the source impedance and Z_o.

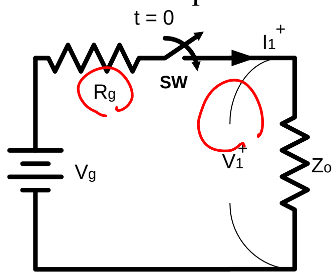
$Z_0 = 50\Omega$

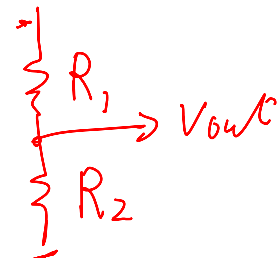
$R_g = 200\Omega$

$Z_L = 100\Omega$ 1V

V_1^+ ?

Voltage divider.





$V_{out} = \dfrac{R_2}{R_1 + R_2}$

R_g=4 Z_o
Z_L=2 Z_o
V_g=1V

$$V_1^+ = I_1^+ Z_o, \quad I_1^+ = \frac{V_g}{R_g + Z_0}, \qquad V_1^+ = \frac{V_g Z_0}{R_g + Z_0} = \frac{1}{5} V_g = 0.2V$$

Final Condition SW : ON very long time

We also need to take a look at a final condition. If the SW is ON for a long time, this corresponds to the DC voltage. The TL has no effect. The voltage on the load is given by

at DC

$\lambda = \infty$

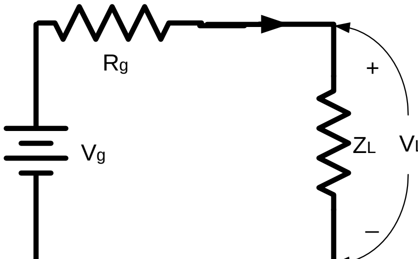
digital multimeter.

Sound speed

~~1500 m/s~~

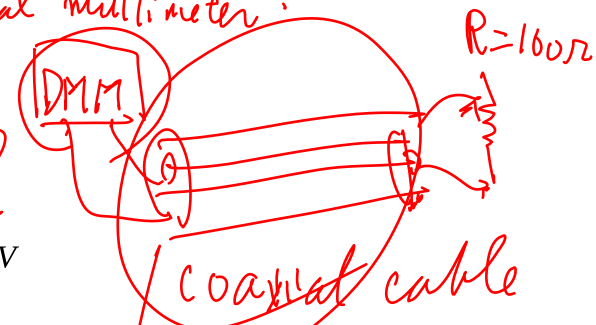
1KHz

20 - 20KHz.



100Ω ?

100Ω



$$V_\infty = \left(\frac{Z_L}{R_g + Z_L} \right) Vg = 0.33V$$

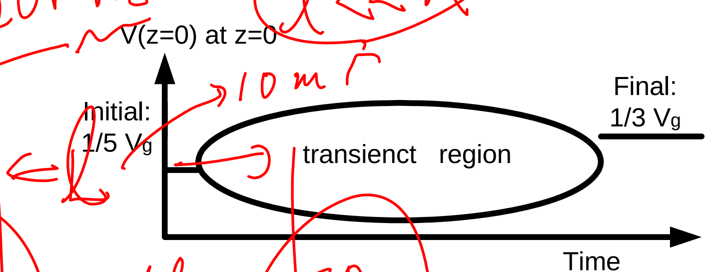
coaxial cable

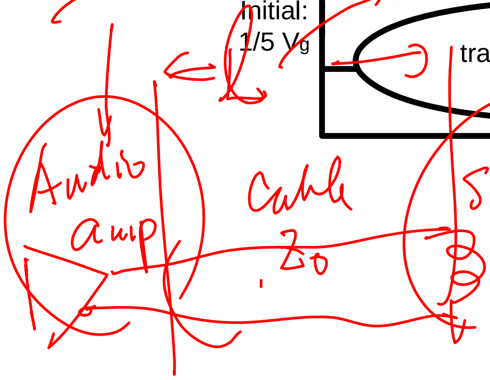
RG58

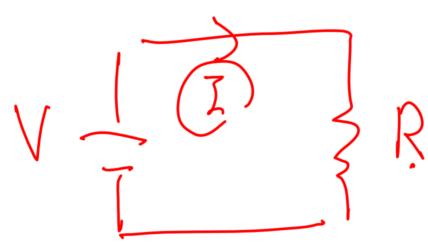
$Z_0 = 50\Omega$.

TV

$l \ll \lambda$







The transient response is, therefore, a time response from the initial condition to the final condition.

Between Initial and Final Conditions (Transient)

The incident voltage reaches the load Z_L at $t=T=l/U_p$ where U_p is the phase velocity on a TL. If the load impedance is not matched to the TL characteristic impedance, part of the incident voltage will be reflected. The polarity (positive or negative) of the reflected voltage depends on the load Z_L. It is positive for $Z_L>Z_o$ and negative for $Z_L<Z_o$. Although we calculate the reflected voltage using the reflection coefficient, what we can observe is the total voltage. The total voltage on a TL is the sum of all incident and reflected voltages which occur up to the observation time.

Load side $\Gamma_L = \dfrac{2Z_0 - Z_0}{2Z_0 + Z_0} = \dfrac{1}{3}$

Source side $\Gamma_g = \dfrac{4Z_0 - Z_0}{4Z_0 + Z_0} = \dfrac{3}{5}$

The first reflected voltage from the load

$$V_1^- = \Gamma_L V_1^+ = \left(\frac{1}{3}\right)\left(\frac{1}{5}\right)V_g = \left(\frac{1}{15}\right)V_g$$

Total voltage is

$$V = V_1^+ + V_1^- = \left(\frac{1}{5} + \frac{1}{15}\right)V_g = \frac{4}{15}V_g \quad (= 0.267V)$$

If the source impedance R_g is matched to a TL ($R_g=Z_o$), the reflected voltage will be absorbed by the source impedance and no signal will be reflected back to the TL when the reflected voltage arrives to the source. However, the reflected voltage which arrives to the source may also see an unmatched impedance ($R_g<>Z_o$). This will create a secondary reflection from the source side which becomes a new incident voltage.

Reflected voltage from the source

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+ = \left(\frac{3}{5}\right)\left(\frac{1}{3}\right)V_1^+ = \left(\frac{3}{15}\right)\left(\frac{1}{5}V_g\right)$$

Total voltage

$$\begin{aligned} V &= V_1^+ + V_1^- + V_2^+ = (1 + \Gamma_L + \Gamma_L\Gamma_g)V_1^+ \\ &= \left(1 + \frac{1}{3} + \frac{3}{15}\right)\left(\frac{1}{5}V_g\right) \\ &= \left(\frac{23}{15}\right)\left(\frac{1}{5}V_g\right) \\ &= \frac{23}{75}V_g \qquad (0.306V) \end{aligned}$$

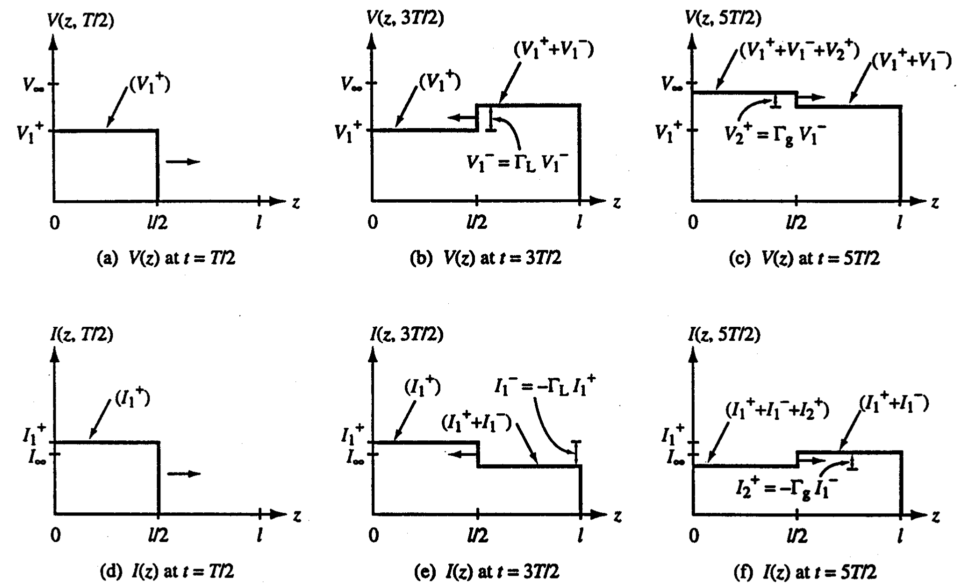


Figure 2-34: Voltage and current distributions on a lossless transmission line at $t = T/2$, $t = 3T/2$, and $t = 5T/2$, due to a unit step voltage applied to a circuit with $R_g = 4Z_0$ and $Z_L = 2Z_0$. The corresponding reflection coefficients are $\Gamma_L = 1/3$ and $\Gamma_g = 3/5$.

This multiple reflection can be shown using the bounce diagrams.

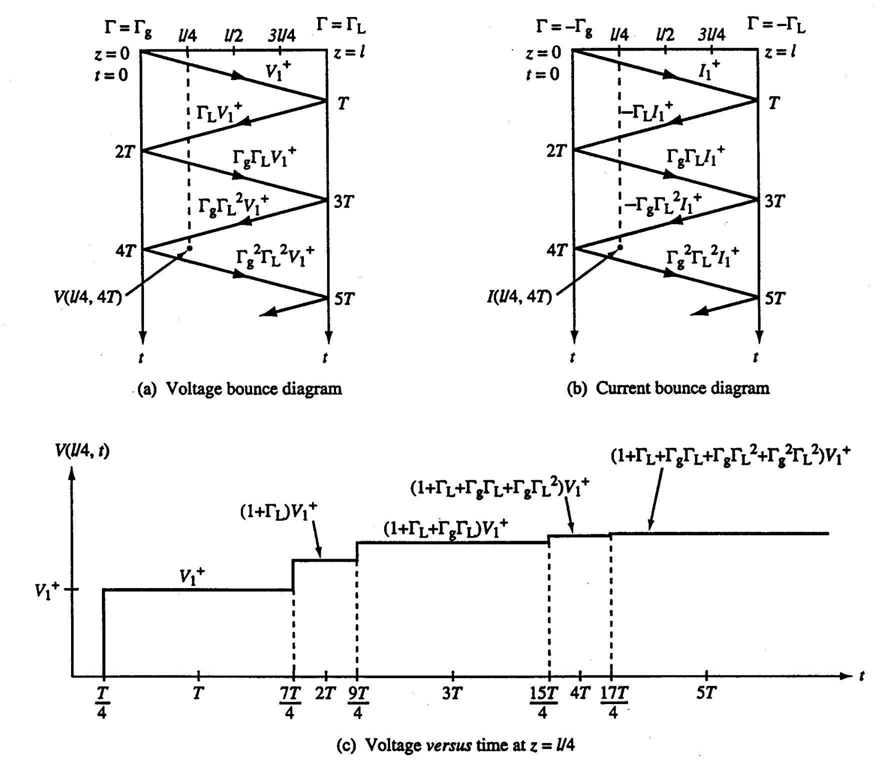


Figure 2-35: Bounce diagrams for (a) voltage and (b) current. In (c), the voltage variation with time at $z = l/4$ for a circuit with $\Gamma_g = 3/5$ and $\Gamma_L = 1/3$ is deduced from the vertical dashed line at $l/4$ in (a).

Useful formula for a infinite series

$$1 + x + x^2 + x^3 .... = \frac{1}{1-x} \qquad |x| < 1$$

$$1 + \Gamma_L + \Gamma_g \Gamma_L + \Gamma_g \Gamma_L{}^2 + (\Gamma_g \Gamma_L)^2 + ... = [1 + \Gamma_g \Gamma_L + (\Gamma_g \Gamma_L)^2 + ..] + [\Gamma_L + \Gamma_g \Gamma_L{}^2 + ...]$$

$$= [1 + \Gamma_L][1 + \Gamma_g \Gamma_L + (\Gamma_g \Gamma_L)^2 ...]$$

Using this we can get

$$V_\infty = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- ...$$

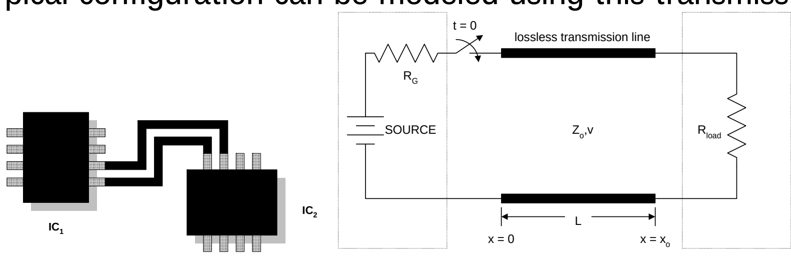
$$= V_1^+ \left[(1 + \Gamma_L)(1 + \Gamma_L \Gamma_g + \Gamma_L^2 \Gamma_g^2 + ...) \right]$$

$$= V_1^+ (1 + \Gamma_L)(1 + x + x^2 + ...), \qquad x = \Gamma_L \Gamma_g$$

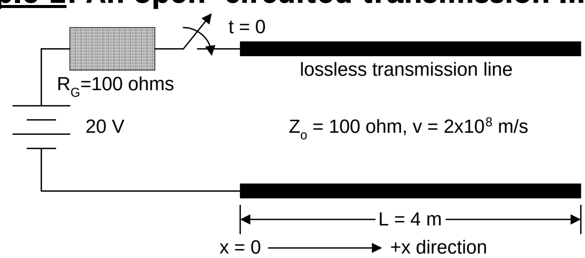
$$V_\infty = V_1^+ \left(\frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_g} \right) = \frac{V_g Z_L}{R_g + Z_L}, \quad 1 + x + x^2 + ... = \frac{1}{1-x}$$

Another Reference Material
Modeling signal propagation between ICs using transmission line

Modern ICs are usually mounted on PCB and interconnected by conducting line of very high conductivity (loss characteristics can be therefore neglected). We start our study with the following example: two ICs are connected in series, with one being master (the *driving* IC – IC_1) and the other slave (the driven IC – IC_2). This typical configuration can be modeled using this transmission line model:



<u>Example 1</u>: An open–circuited transmission line ($Z_{load} \to \infty$)



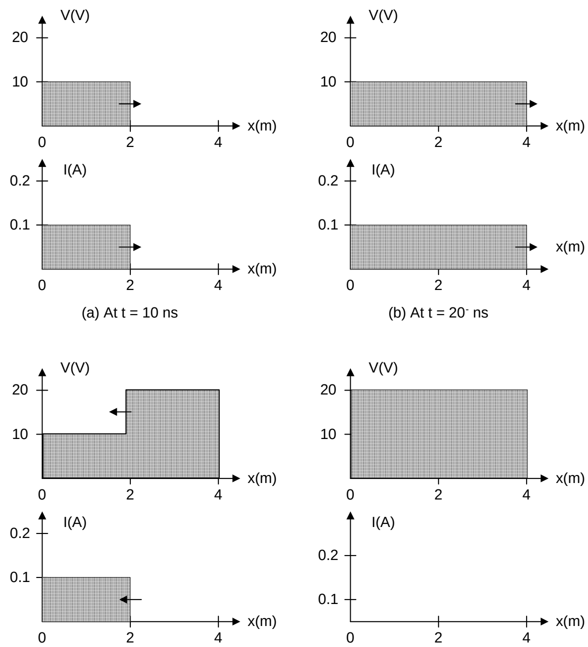
Describe in words and space-diagrams what happens as signal propagates from voltage source to the far-end of the transmission line.

<u>Solution</u>:

Consider the case of a 20 V battery with R_G = 100 ohms connected to an open–circuited, lossless transmission line. This situation, with specific values of L, v and Z_o, is shown in the figure above. With the initially open switch closed at t = 0, the voltage at the input to the line immediately becomes 10 V. This occurs because at the first instant, the dc source has no indication that the line is not infinite in length and hence "sees" an input impedance Z_o = 100 ohms. Thus at $t = 0^+$ (that is, immediately after closing the switch), the current and voltage at the input to the line are $20/(R_G + Z_o)$ = 0.10 A and 10 V, respectively. These values remain constant until the battery has some indication (via a reflected wave) that the line is not infinite in length. With the velocity given as 2 x 10^8 m/s, it takes 10 ns for **V** and **I** to travel halfway down the 4 m line. This situation is shown in part (a) of Fig. 1. Part (b) shows the waves at $t = 20^-$ ns (that is, slightly less than 20 ns). When the waves arrive at the open circuit, something must happen since two contradictory impedance requirements exist. First, the **V**/**I** ratio for the traveling wave must be Z_o = 100 ohms. On the other hand, Ohm's law at the open–circuited end of the line requires an infinite impedance since current must be zero. The creation of reflected waves (**V_-, I_-**) allows both of these requirements to be satisfied. Thus at the load end (x = 4 m, corresponding t = 20 ns),

$$\mathbf{V_{load}} = \mathbf{V_+} + \mathbf{V_-}$$

$$\mathbf{I_{load}} = \mathbf{I_+} - \mathbf{I_-}$$



(a) At t = 10 ns

(b) At t = 20^- ns

(c) At t = 30 ns

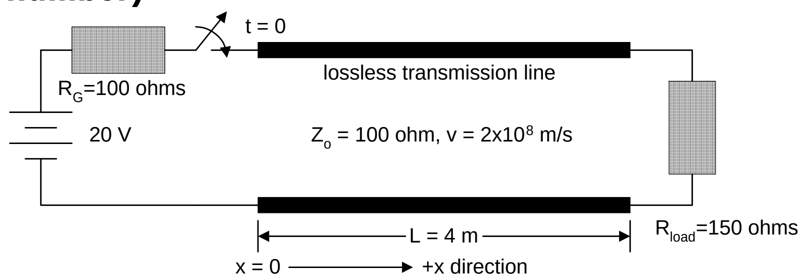
(d) At t = 40 ns

Fig. 1

The condition $\mathbf{I_{load}}$ = 0 requires $\mathbf{I_-}$ and $\mathbf{I_+}$ = 0.10 A. Also, with $\mathbf{V_+} = \mathbf{I_+ Z_o}$ and $\mathbf{V_-} = \mathbf{I_- Z_o}$, $\mathbf{V_-} = \mathbf{V_+}$ = 10 V. Therefore, $\mathbf{I_{load}}$ = 0 A and $\mathbf{V_{load}}$ = 20 V at the load end.

The open–circuit condition at the load end creates reflected voltage and current waes of 10 V and 0.10 A, respectively. These waves travel in the negative x direction with the same velocity as the incident (transmitted) waves. Part (c) and (d) of Fig. 1 show the resultant voltage and current (due to the sum of + and – waves) at t = 30 ns and 40 ns. As the wavefront of the 10 V, 0.10 A reflected waves moves to the left, it leaves behind a net voltage of 20 V and a net current of zero. Since R_G = 100 ohms, both Ohm's law and the condition that $\mathbf{V_-}/\mathbf{I_-}$ = 100 ohms are satisfied at t = 40 ns, and hence no reflections are required at the generator end. The process thus ends and a steady state is achieved with **V** = 20 V and **I** = 0 A everywhere on the transmission line. The time–flow plots of these incident and reflected signals as a function of distance and time are shown on the next page.

Example 2: A resistively terminated transmission line (Z_{load} : real number)



Describe in words and space-diagrams what happens as signal propagates from voltage source to the far-end of the transmission line.

Solution:

Consider now the case of a finite length transmission line terminated with a pure resistance. This situation is shown above, where R_{load} (= 150 ohms) is the terminating or load resistance. As before, closing the switch initiates a 10 V, 0,10 A forward traveling wave. At t = 20 ns, the wave arrives at the load end. Since $R_{load} \neq Z_o$, Ohm's law can only be satisfied by assuming reflected waves. Thus at z = 4 m, $\mathbf{V_{load}} = \mathbf{V_+} + \mathbf{V_-}$ and $\mathbf{I_{load}} = \mathbf{I_+} - \mathbf{I_-} = (\mathbf{V_+} - \mathbf{V_-})/Zo$. Ohm's law requires $\mathbf{V_{load}}/\mathbf{I_{load}} = R_{load}$ and hence

$$R_{load} = Z_o(\mathbf{V_+} + \mathbf{V_-}) / (\mathbf{V_+} - \mathbf{V_-}) = Z_o[\ 1 + (\mathbf{V_-}/\mathbf{V_+})\ ] / [\ 1 - (\mathbf{V_-}/\mathbf{V_+})\ ]$$

$$= Z_o(1 + \Gamma_{\mathbf{load}}) / (1 - \Gamma_{\mathbf{load}})$$

Solving for the load reflection coefficient yields

$$\Gamma_{\mathbf{load}} = (R_{load} - Z_o) / (R_{load} + Z_o)$$

(compare this with $\Gamma_{\mathbf{load}} = (\ \mathbf{Z_{load}} - Z_o\ ) / (\ \mathbf{Z_{load}} + Z_o\ ) = \mathbf{V_-}(x=0) / \mathbf{V_+}(x=0)$
on page 3 of this handout).

For resistively terminations, $\Gamma_{\mathbf{load}}$ is real and can take on any value between –1 and +1. If R_{load} = 0 (short circuit), $\Gamma_{\mathbf{load}}$ = –1, while if $R_{load} = \infty$ (open circuit, the previous case), $\Gamma_{\mathbf{load}}$ = +1.

In the present case for R_{load} = 150 ohms, Γ_{load} = 0.2. With the forward wave equal to 10 V and 0.10 A, the reflected voltage and current are 10 V $\times$ 0.2 = 2 V and 0.10 A $\times$ 0.2 = 0.02 A, respectively. Parts (a) and (b) of Fig. 2 shows the voltage and current along the line at t = 10 ns and 30 ns. At t = 10 ns, only the forward traveling waves exist, having arrived only at the halfway point of the 4 m line. At t = 30 ns, the reflected waves have been generated and have traveled halfway back toward the generator end of the line. At t = 40 ns (not shown), the reflected waves arrive at the input and the resultant voltage and current everywhere along the line become 12 V and 0.08 A. Since $R_G = Z_o$, no reflection is required at the generator end and the steady state is achieved after 40 ns. Note that the final values (12 V and 0.08 A) are those expected from a dc analysis of the circuit.

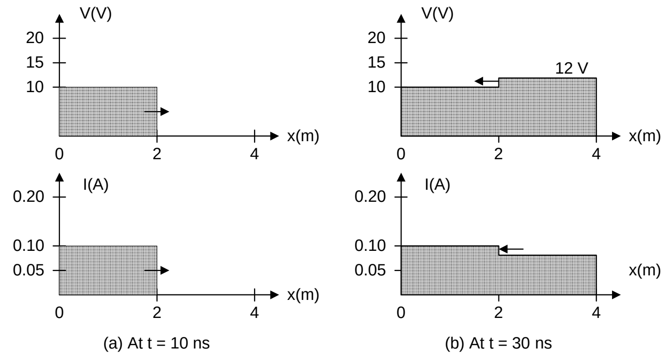


Fig. 2

<u>Example 3</u>: Multiple reflections on a transmission line

From the above cases, it is clear that when $R_G = Z_o$, the steady state is achieved after one round trip (40 ns, in our example). On the other hand, if $R_{load} = Z_o$, the steady state occurs after an *one-way* trip (20 ns, in our example). Let us now explore the situation when neither R_G nor R_{load} is equal to the characteristic impedance Z_o. The analysis will show that reflections occur at both ends of the line and the steady start values are approached only as t becomes *infinite*!

Describe in words and space-diagrams what happens as signal propagates from voltage source to the far-end of the transmission line.

<u>Solution</u>:

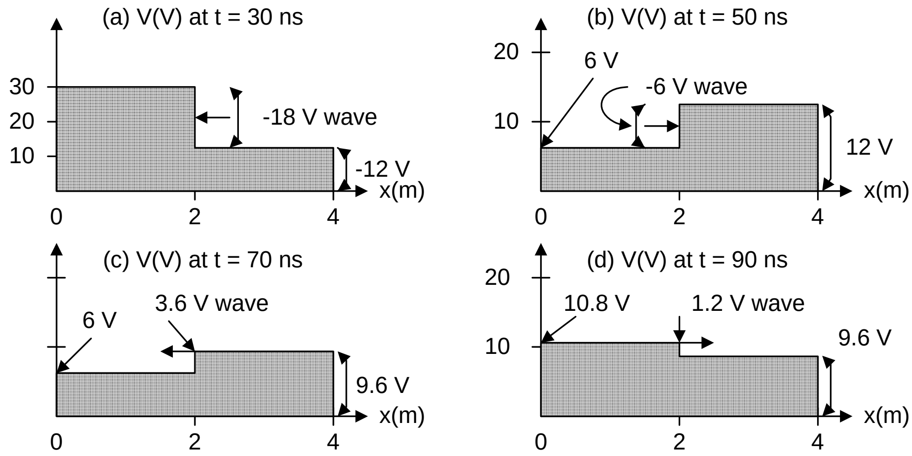
As a specific example, consider the circuit above, where R_G = 200 ohms, R_{load} = 25 ohms, and Z_o = 100 ohms. When the switch is closed at t = 0, the 90 V source sees 200 ohms in series with the characteristic impedance of the line. Therefore, the current and voltage at the input end of the line (x = 0) are initially $\mathbf{I_+}$ = 90/300 = 0.3 A and $\mathbf{V_+}$ = $\mathbf{I_+}Z_o$ = 30 V. After 20 ns, the $\mathbf{V_+}$ and $\mathbf{I_+}$ waves arrive at the load end where the reflection coefficient $\Gamma_{\mathbf{load}}$ = (25 – 100)/(25 + 100) = –75/125 = –0.6 and hence $\mathbf{V_-}$ = $\Gamma_{\mathbf{load}} \times \mathbf{V_+}$ = –18 V and $\mathbf{I_-}$ = $\Gamma_{\mathbf{load}} \times \mathbf{I_+}$ = –0.18 A. At the end of 3o ns, the voltage between x = 2 m and x = 4 m is reduced to 30 – 18 = 12 V, while the current has increased to 0.3 + 0.18 = 0.48 A. The progress of the voltage wave along the line is shown in Fig. 3 for t = 30, 50, 70, and 90 ns.

Let us observe the voltage wave as time marches on. At the end of 40 ns, the –18 V wave arrives at the input where it sees an impedance R_G = 200 ohms. Since $R_G \neq Z_o$, a reflection occurs at the generator end. By analogy with $\Gamma_{\mathbf{load}}$, the generator reflection coefficient $\Gamma_{\mathbf{G}}$ is given by

$$\Gamma_{\mathbf{G}} = (R_G - Z_o) / (R_G + Z_o)$$

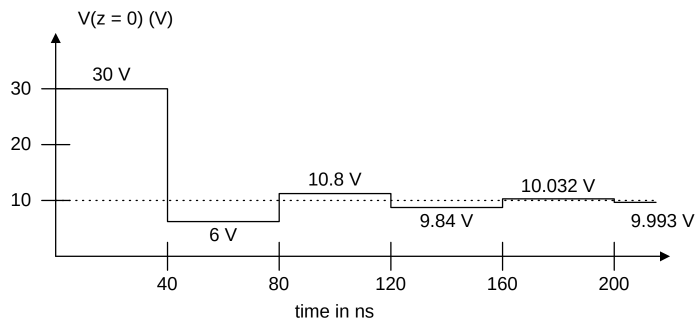
For R_G = 200 ohms, $\Gamma_{\mathbf{G}}$ = 1/3 and hence a –6 V wave is reflected towards the load end. At t = 50 ns, it has progressed halfway down the line, leaving behind it a voltage of (30 – 18 – 6) = 6 V. This is shown in part (b) of the figure. At t = 60 ns, the –6V wave arrives at the load which generates a reflected wave of value (–6) $\times \Gamma_{\mathbf{load}}$ = +3.6 V. The situations at 70 and 90 ns are also shown in the figure. Note that at t = 90 ns, another forward traveling wave exists having a value (+3.6) $\times \Gamma_{\mathbf{G}}$ = +1.2V. This process continues indefinitely with the amplitude of the re-reflected waves getting smaller and smaller (due to energy dissipation of the resistors). A plot of voltage versus time at any fixed point on the line would show that, in the limit, the voltage becomes the expected dc value (namely, $90R_{load}/(R_G + R_{load})$ = 10 V). Such a plot at x = 0, the input, is shown in Fig. 4. Every step in voltage represents the arrival and generation of the reflected waves at the input. After five round-trip (200 ns), the voltage is within 0.10 percent of the steady-state value.

It is interesting to note that the voltage shown in Fig. 4 is oscillatory as it approaches its final value. The period of this ringing effect is 80 ns (twice the round-trip time) and hence its reciprocal is the natural resonant frequency of the circuit, namely, 12.5 MHz. Since v = 2×10^8 m/s, this means that the line is $\lambda/4$ long at the resonant frequency. Thus we see that by connecting a dc source to a transmission line, high frequency oscillations are possible. In a PCB system, the emitted power from these high-frequency oscillations, if left unchecked, will result in interference to other devices, impairing the performance of the overall system.



(a) V(V) at t = 30 ns
30
20
10
-18 V wave
-12 V
x(m)
0
2
4
(b) V(V) at t = 50 ns
20
10
6 V
-6 V wave
12 V
x(m)
0
2
4
(c) V(V) at t = 70 ns
6 V
3.6 V wave
9.6 V
x(m)
0
2
4
(d) V(V) at t = 90 ns
20
10
10.8 V
1.2 V wave
9.6 V
x(m)
0
2
4

Fig. 3



V(z = 0) (V)
30
20
10
30 V
10.8 V
10.032 V
6 V
9.84 V
9.993 V
40
80
120
160
200
time in ns

Fig. 4

2. Space-Time Representation of Signals

The space-time diagram is a graphical aid in determining the voltage and current as a function of either time or position along the line. Fig. 5 shows the diagram for the circuit in Example 3. The abscissa indicates position along the line and the ordinate represents the time scale, t = 0 being the moment that the switch is closed. For reference, the values of Γ_G and Γ_{load} are given at the top of the diagram. The lines sloping downward and to the right represent forward traveling waves, while those sloping down and to the left represent reverse waves. The voltage and current values for the particular wave are shown above and below the sloping line. As explained, the load end creates reflections equal to Γload of the arriving wave. Generator reflections are equal to Γ_G times the value of the wave arriving at the generator end.

To illustrate, Fig. 5 will be used to determine the voltage and current at x = 2 m. Each intersection of a sloping line with the interval x = 2 m line represents the arrival of a wavefront. For t < 10 ns, no intersection exists and hence both **V** and **I** are zero. For 10 < t < 30 ns, there is one intersection which means **V** = 30 V and I = 0.30 A. For t > 30 ns, the voltage is the sum of all the forward and reverse waves that have passed the x = 2 m location. For example, at t = 80 ns, **V** = 30 – 18 – 6 + 3.6 = 9.6 V. The current may be determined in a similar manner except that current values associated with reverse waves must be subtracted from those associated with the forward waves. For example, at t = 80 ns, **I** = 0.30 – (–0.18) + (–0.06) + (–0.06) – (+0.036) = 0.384 A. The diagram may also be used to determined voltage and current versus x for a fixed time by drawing a horizontal line corresponding to the particular value of time. The sum of voltages above the line corresponds to the voltage at that point on the line. The same applies to the current except that, as before, reverse–traveling current waves must be subtracted from forward–traveling current waves.

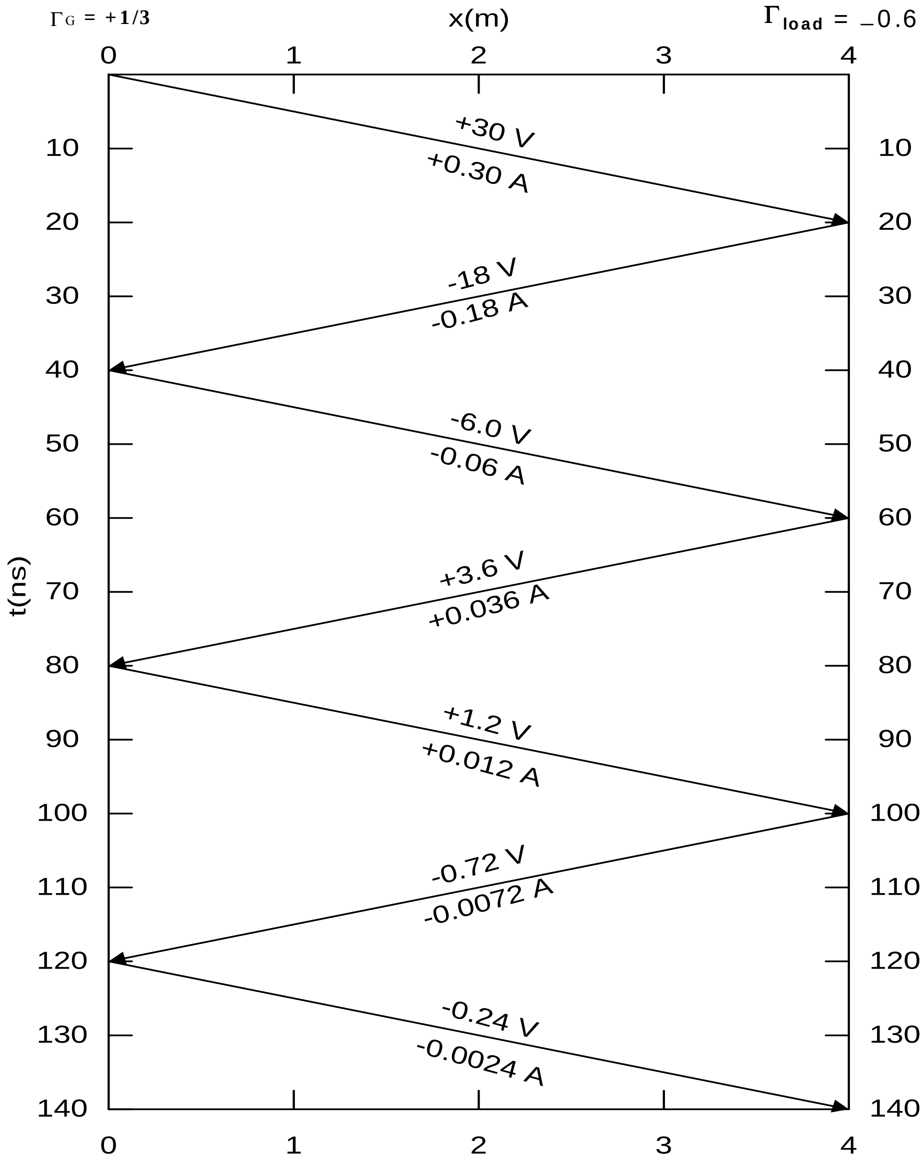
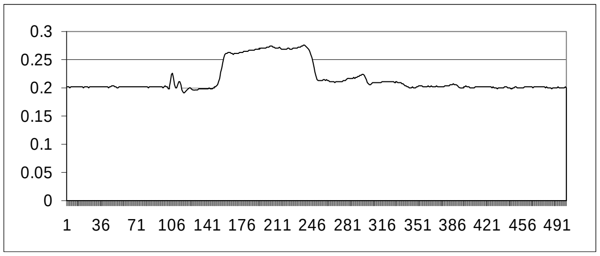
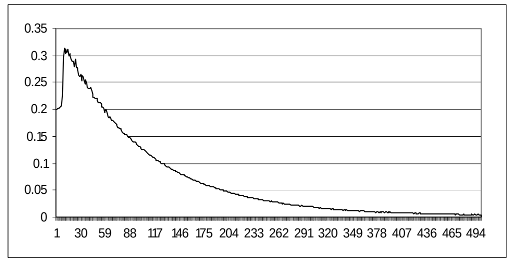

ΓG = +1/3
x(m)
Γload = −0.6
0
1
2
3
4
t(ns)
10
20
30
40
50
60
70
80
90
100
110
120
130
140
+30 V
+0.30 A
-18 V
-0.18 A
-6.0 V
-0.06 A
+3.6 V
+0.036 A
+1.2 V
+0.012 A
-0.72 V
-0.0072 A
-0.24 V
-0.0024 A

Fig. 5

Expected responses (different TDR is used for these case)

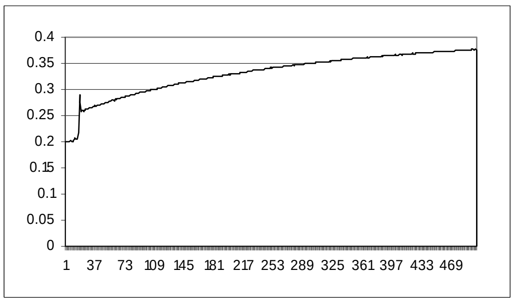


a) Unmatched Transmission Line

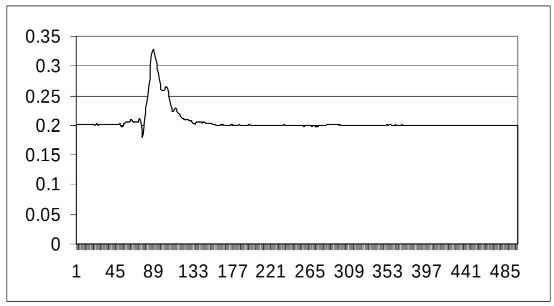


(b) Unknown1

(1 data point = 10ps)

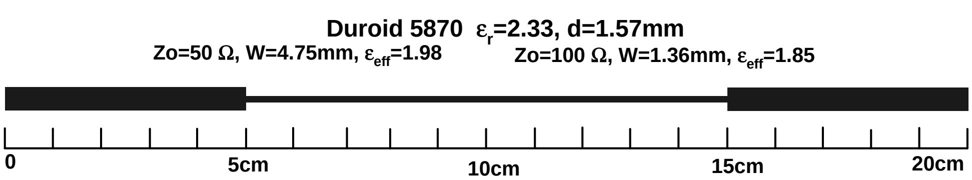


(c) Unknown 2



(d) Unknown 3

Pictures of devices



Test PCB

Unknown device 1

Unknown device 2

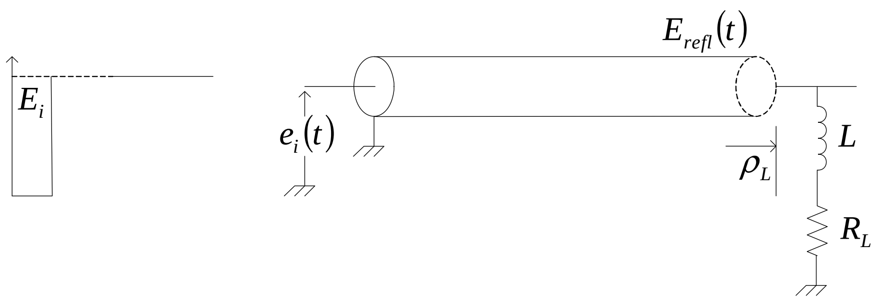
Unknown device 3

5. Laplace transform analysis of a TL terminated with a complex load impedance **10/27/10**

In the previous section, we studied the reflection of a unit step function from the unmatched load which is purely resistive. The rise-time (waveform) of the reflected signal does not change if the unmatched load is resistive. In reality, however, the input/load impedances of the digital circuit contain a reactive element such as a parasitic capacitance and inductance. In this section, therefore, we will study the time-domain analysis of reflected signal from a complex load impedance. The analysis is based on Laplace transform . We express the input waveform and complex load impedance using Laplace transform and calculate the reflected voltage. By taking the inverse Laplace transform, we can derive the time-domain response. The waveform of the reflected signal shows the distinctive characteristics depending on the load type. This can be used for inferring the load type. In addition, we will show how to obtain the values of each element using different techniques.

5-1. Unit step function as an input (Reflection case)

We have a L-R series load attached to a lossless TL which does not distort the waveform. The input signal is a unit step function. To simplify the analysis, we will neglect the delay due to a TL.



Using the Laplace transform, the load impedance is given by $Z_L = R_L + sL$.

The reflection coefficient at the load can be expressed as

$$\rho_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{s + \dfrac{R_L - Z_0}{L}}{s + \dfrac{R_L + Z_0}{L}}$$

The input voltage is the unit step function and is given by

$$e_i(t) = u(t)$$

The Laplace transform of this is

$$E_i(s) = \frac{1}{s}$$

where Laplace transform of $[u(t)] = \frac{1}{s}$

The reflected voltage from the load is given by

$$E_{refl}(s) = E_i(s)\rho_L(s) = \left(\frac{1}{s}\right)\left(\frac{s + \frac{R_L - Z_0}{L}}{s + \frac{R_L + Z_0}{L}}\right)$$

Now we use the inverse Laplace transform to get the time-domain response $E_{refl}(t)$

We need to separate the above equation into simple terms which represent each response such as a unit step function u(t). We can write the term as

$$\frac{s+A}{s(s+B)} = \frac{C_1}{s} + \frac{C_2}{(s+B)} = \frac{C_1(s+B) + C_2 s}{s\ (s+B)}$$

Therefore

$$C_1 + C_2 = 1$$
$$C_1 B = A$$

$$C_1 = \frac{A}{B} = \frac{R_L - Z_0}{R_L + Z_0}$$
$$C_2 = 1 - C_1 = \frac{2Z_0}{R_L + Z_0}$$

Now we take the inverse Laplace transform and obtain

$$\left[\frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

where $\frac{1}{S+B} \rightarrow e^{-Bt}$

The final expression of the reflected signal is

$$E_{refl}(t) = \left[\frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

The total voltage is the sum of incident and reflected at the load. Therefore, we have the load voltage of

$$E_{total}(t) = E_{inc} + E_{ref} = \left[1 + \frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

It is easy to see the peak voltage occurs at t=0 and the value is given by

$$E_{total}\left(t=0\right)=\left[\frac{2R_L+2Z_0}{\left(R_L+Z_0\right)}\right]=2$$

This is expected because the inductance is an open circuit at t=0 and the total voltage becomes twice the incident voltage.

The final voltage is given by setting $t \to \infty$. The value is

$$E_{total}\left(t=\infty\right)=\left[\frac{2R_L}{\left(R_L+Z_0\right)}\right]$$

The transient region is given by the exponential decay $e^{-\left(\frac{R_L+Z_0}{L}\right)t}$.
If we take a natural log of this response, we get a line as a function of time.

$$\log[e^{-\left(\frac{R_L+Z_0}{L}\right)t}]=-\left(\frac{R_L+Z_0}{L}\right)t=-mt$$

The above expression shows the slope of this line *m* is proportional to L. Because we can find R_L from the $t=\infty$ data, we should be able to obtain the value of L from the slope *m*. This process can be used with the TDR response to obtain the values of L and R_L.

If the load is given by the other combinations of L, C and R, we need to replace Z_L by

Parallel L and R

$$Z_L = R_L // sL = sR_L L/(R_L+sL)$$

Series C and R

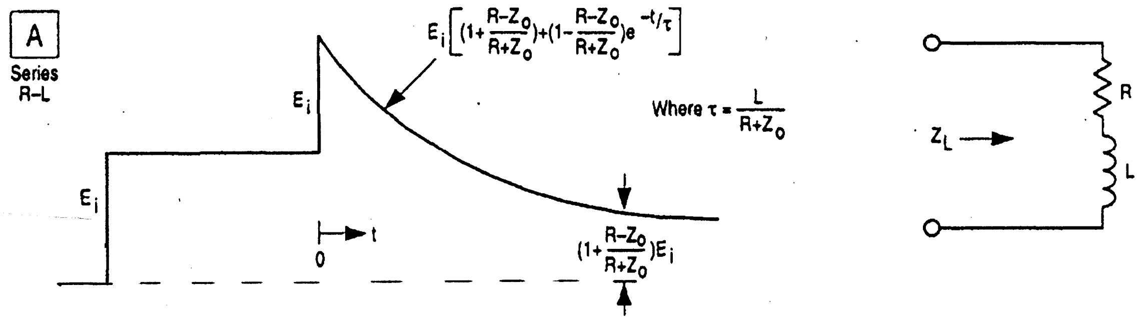
$$Z_L = R_L + (1/sC)$$

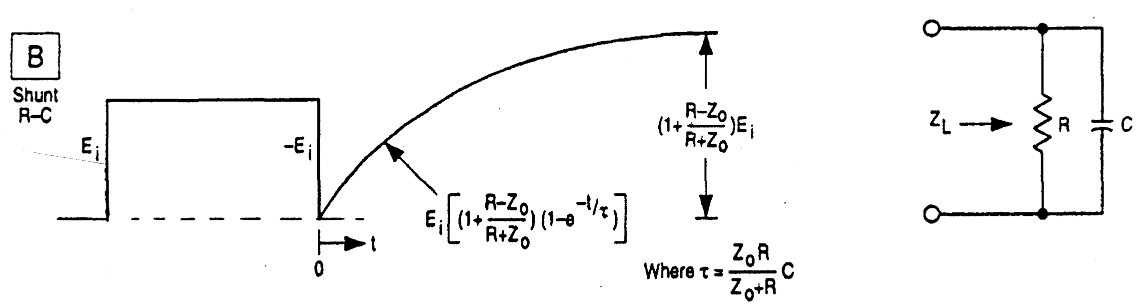
Parallel C and R

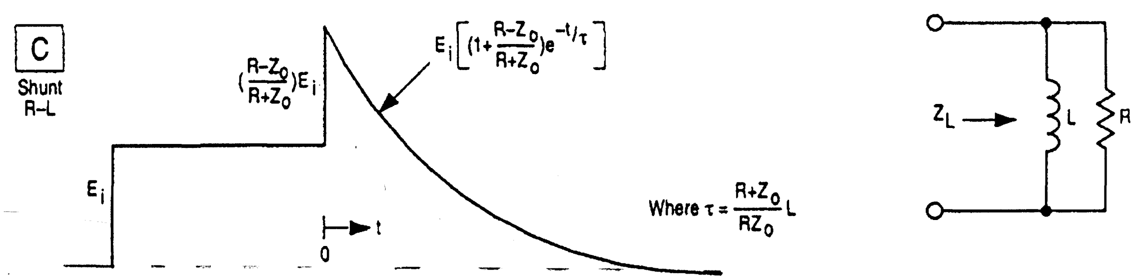
$$Z_L = R_L // sC = R_L/(1+sCR_L)$$

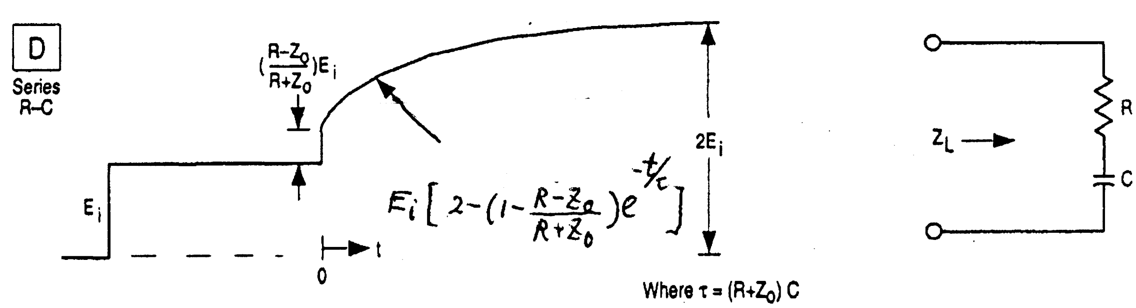
The expected waveforms and responses are shown below. In all cases, the values of L and C can be estimated by taking a natural log of the time-domain responses. However, it is also clear that neither the initial peak voltage nor the final voltage can be used for estimating L and C. This limitation is due to the use of a ideal unit step function.
In the next section, we will study the finite rise-time case and show that the peak voltage value can be used for estimating the value of L when the inductance is present.

Expected TDR responses from different complex loads









Estimation of complex load values for the unit step function excitation

Assume we have a series L and R circuit and we get

$$E_{total}(t) = C_1\left[1 + \frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

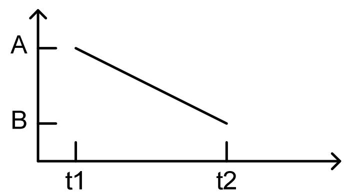
where C_1 is a constant related to the initial peak voltage of TDR.

We can assume E_{total} is also our measured data. We can find R_L from either the initial or final condition. To find L, we set

$$e^{-\left(\frac{R_L + Z_0}{L}\right)t} = [(E_{total}(t)/C_1) - (1 + \frac{R_L - Z_0}{(R_L + Z_0)})]\left[\frac{(R_L + Z_0)}{2Z_0}\right] = C_2$$

$$\ln(e^{-\left(\frac{R_L + Z_0}{L}\right)t}) = -\left(\frac{R_L + Z_0}{L}\right)t = -mt = \ln(C_2)$$

where m is a slope of $y=-mt$ line of the experimental data. C_2 must be positive. C_2 has a linear section and you can find the slope as $m=(A-B)/(t2-t1)$. Although E_{total} is your measured data, the constant term in C_2 is not important. You can use your experimental data as C_2 in the analysis as shown below.



Linear section of C_2 is shown.

When we have R and C, we have (1-exp(-mt)) response. In this case you cannot take ln(experimental data) to get a slope. You need to change the data to get the form exp(-mt)=C_x equation.

Assume we have a parallel R and C load.

$$E_{total}(t) = C_1\left[(1 + \frac{R_L - Z_0}{(R_L + Z_0)})(1 - e^{-mt})\right] u(t)$$

$$e^{-mt} = 1 - E_{total}(t)/[C_1\left[(1 + \frac{R_L - Z_0}{(R_L + Z_0)})\right]] = C_3$$

where C_1 is a constant related to the initial peak voltage of TDR. In this expression E_{total} is your measured data. Also C_3 must be positive to use ln() function.

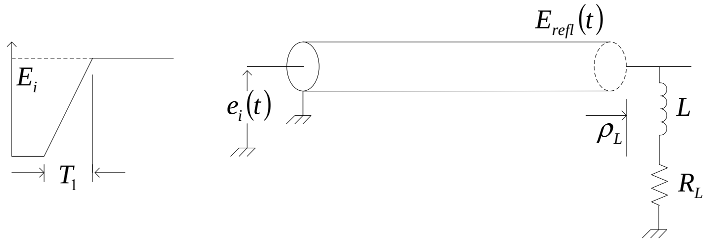
Another approach is the "trial and error" or "supervised parameter estimation". Assume you already know C_1 and R_L in the following equation and the only unknown is L.

$$E_{total}(t) = C_1\left[1 + \frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

Set L to be a certain value then calculate $E_{total}()$ and compare it with experimental data. Time t=0 corresponds to the start of the reflected voltage. The initial voltage exists for t<0, shown as E_i. If the tail is too long, L is too large. If the tail is too short, L is too small. You may use the "binary search" method to find the optimum value of L.

5-2. Finite rise-time input (Reflection case) (This is more realistic model.)

In 5-1, we studied the simple unit step function responses. The practical digital circuits, however, have a finite rise-time signal and complex load. In this section, we will use the Laplace transform technique to analyze both finite rise-time signal and reflection from a complex load. We assume the TL is lossless. The lossy TL case which will create dispersion (distortion of the waveform) will be analyzed later.



Let assume the input pulse rise-time (10-90%) is specified as t_r =80ps. Using the straight line approximation, the total transient time is given by

$$T_1 \approx \frac{t_r}{0.8} = 100\,ps$$

The slope of the input signal (rise) is then $m = \dfrac{E_i}{T_1}$.

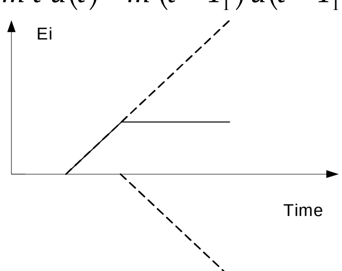
We use the same complex load given by $Z_L = R_L + sL$

The reflection coefficient at the load can be expressed as

$$\rho_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{s + \dfrac{R_L - Z_0}{L}}{s + \dfrac{R_L + Z_0}{L}}$$

The input voltage can be decomposed into two terms is given by

$$e_i(t) = m\,t\,u(t) - m\,(t - T_1)\,u(t - T_1)$$



We take Laplace transform of $e_i(t)$ and obtain

$$E_i(s) = \frac{m}{s^2}\left(1 - e^{-T_1 s}\right)$$

$e^{-T_1 s}$: delay

where $u(t)=\dfrac{1}{s}$

$$tu(t)=\frac{1}{s^2}$$

The reflected voltage from the load is given by

$$E_{refl}(s)=E_i(s)\rho_L(s)=\left(\frac{m}{s^2}\right)\left(1-e^{-T_1 s}\right)\left(\frac{s+\frac{R_L-Z_0}{L}}{s+\frac{R_L+Z_0}{L}}\right)$$

Now we use an inverse Laplace transform to get the time-domain response $E_{refl}(t)$

$$E_{refl}(s)=\left(\frac{m}{s^2}\right)\left(\frac{s+\frac{R_L-Z_0}{L}}{s+\frac{R_L+Z_0}{L}}\right)\text{-}\left(\frac{m}{s^2}\right)e^{-T_1 s}\left(\frac{s+\frac{R_L-Z_0}{L}}{s+\frac{R_L+Z_0}{L}}\right)$$

We need to separate the above equation into simple terms which represent each response such as a unit step function u(t). We can write the first term as

$$\frac{s+A}{s^2(s+B)}=\frac{C_1}{s}+\frac{C_2}{s^2}+\frac{C_3}{(s+B)}=\frac{C_1 s(s+B)+C_2(s+B)+C_3 s^2}{s^2(s+B)}$$

Therefore

$$C_1+C_3=0$$
$$C_1B+C_2=1$$
$$C_2B=A$$

$$C_2=\frac{A}{B}=\frac{R_L-Z_0}{R_L+Z_0}$$
$$C_1=\frac{1-C_2}{B}=\frac{2Z_0L}{(R_L+Z_0)^2}$$
$$C_3=-C_1$$

Take an inverse Laplace transform and obtain

1st term

$$\left[\frac{2Z_0L}{(R_L+Z_0)^2}+\left(\frac{R_L-Z_0}{R_L+Z_0}\right)t-\frac{2Z_0L}{(R_L+Z_0)^2}e^{-\left(\frac{R_L+Z_0}{L}\right)t}\right]u(t)$$

$$\frac{1}{S+B}\rightarrow e^{-Bt}$$

We can obtain the 2nd term using the same method.

Then the final expression of the reflected voltage is

$$E_{refl}(t)=\left[\frac{2Z_0L}{(R_L+Z_0)^2}+\left(\frac{R_L-Z_0}{R_L+Z_0}\right)t-\frac{2Z_0L}{(R_L+Z_0)^2}e^{-\left(\frac{R_L+Z_0}{L}\right)t}\right]m\ u(t)$$
$$-\left[\frac{2Z_0L}{(R_L+Z_0)^2}+\left(\frac{R_L-Z_0}{R_L+Z_0}\right)(t-T_1)-\frac{2Z_0L}{(R_L+Z_0)^2}e^{-\left(\frac{R_L+Z_0}{L}\right)(t-T_1)}\right]m\ u(t-T_1)$$

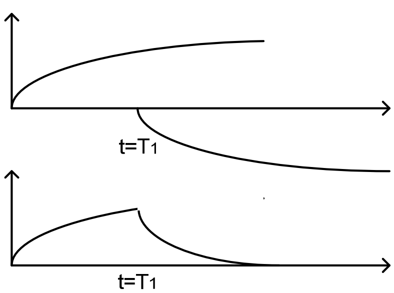
The total voltage is the sum of incident and reflected at the load. Therefore, we have the load voltage of

$$E_{total}(t)=E_{inc}+E_{ref}$$

This should provide the complete voltage response.

It is often important to estimate the maximum reflected voltage. As we found in 5-1, the maximum value is $2E_{inc}$ if the input is a unit step function.

**** This is not correct
To obtain the maximum reflected voltage, we take a derivative of $E_{refl}(t)$ with respect to t and set it to zero. This gives us the condition that the maximum voltage occurs at $t=T_1$.
*****
To simplify, assume $R_L=Z_0$ and plot $E_{refl}(t)$. We can show the peak occurs at $t=T_1$.



Therefore

$$E_{refl\max}=E_{refl}(t=T_1)$$

For a special case of $R_L=Z_0$, the maximum voltage becomes

$$E_{refl}(t=T_1)=\frac{mL}{2Z_0}\left[1-e^{-\left(\frac{2Z_0}{L}\right)T_1}\right]$$

Although L is contained in two places, this expression can be used for estimating the inductance L.

If we approximate exp() as $e^x \sim 1+x+x^2/2$ ($e^x \sim 1+x$ does not work in this case), we get

$$E_{refl}(t=T_1) \approx \frac{mL}{2Z_0}\left[1-[1-\left(\frac{2Z_0}{L}\right)T_1+[\left(\frac{2Z_0}{L}\right)T_1]^2/2\right] = mT_1\left[1-\left(\frac{Z_0}{L}\right)T_1\right]$$

Then the approximate value of L is

$$L \approx Z_0T_1/[1-E_{refl}(t=T_1)/mT_1]$$

Another approach is to calculate the area of the reflected voltage which is the same as integration over the reflected voltage. If $R_L = Z_0$, the final value is the same as the incident voltage and the effect of the incident voltage can be neglected. However, if $R_L \neq Z_0$, the integration over the initial and final values must be carefully done.

Assume $R_L = Z_0$. The integration of the reflected voltage is given by

$$\int_0^\infty E_{refl}(t)dt = \int_0^\infty \frac{mL}{(2Z_0)}\left[1-e^{-\left(\frac{2Z_0}{L}\right)t}\right]dt - \int_{T_1}^\infty \frac{mL}{(2Z_0)}\left[1-e^{-\left(\frac{2Z_0}{L}\right)(t-T_1)}\right]dt = \frac{mLT_1}{2Z_0}$$

This should be set equal to the measured area under the reflected voltage. Then L can be estimated as

$$L = \frac{2Z_0}{mT_1}\int_0^\infty E_{refl\_measured}(t)dt$$

5.3 Estimation of the inductance *L* (Examples using the next figure)

(a) Based on the peak value

$T_1 \approx \frac{t_r}{0.8} = 100ps$ $\qquad$ t_r : 10 to 90%, T_1 : total risetime

We want to find the parasitic inductance of the load.
The peak reflected voltage is given by = 0.44V

$m = \frac{0.94}{100ps}$ $\qquad$ 0.94V: initial voltage

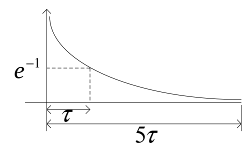
From $\qquad 0.44 = \frac{mL}{2Z_0}\left[1-e^{-\left(\frac{2Z_0}{L}\right)T_1}\right],\ Z_0 = 50\Omega$

We can find L to be

$$L = 6nH$$

If we use the approximation $L \approx Z_0T_1/[1-E_{refl}(t=T_1)/mT_1]$, we get $L \approx 10nH$ which is substantially different from the previous estimate. In this case the argument of the exp() is close to -1 and the approximation based on the 3 terms is not sufficient.

(b) Based on e^{-1} value. If the time-domain response is close to the exponential function and the inductance can also be estimated from the decay time as shown below.



$$\tau = \frac{L}{2Z_0} \quad \text{where} \quad e^{-\frac{t}{\tau}}$$

$$L = (\tau)(2Z_0)$$

If the decay time is $5\tau \sim 0.3ns$, then

$$L \approx (60ps)(100) \approx 6nH$$

This value is close to an estimate based on the peak value.

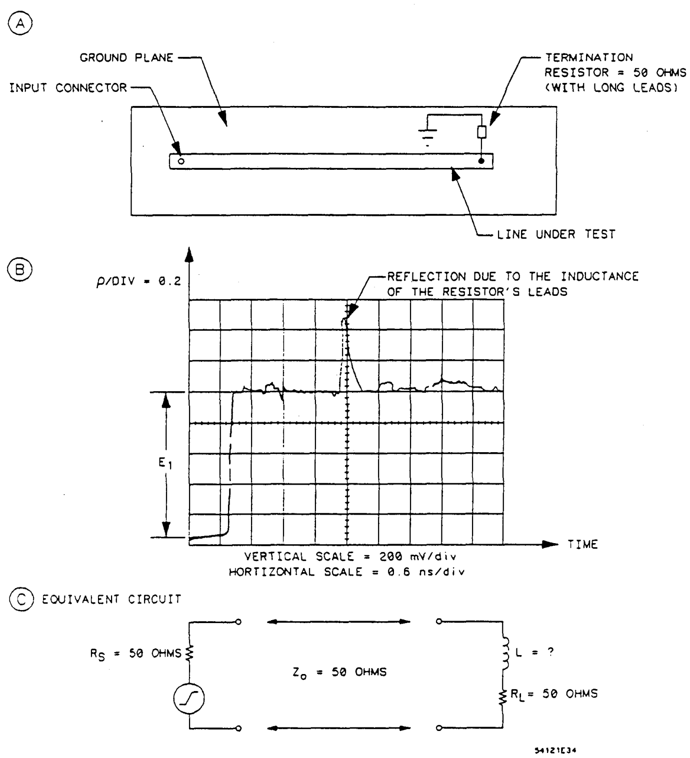


Figure E-10. Effects Due to Termination Resistor Leads

These two methods to estimate L have serious problems. For example, the max value of E_{refl} depends on the rise-time t_r in the first approach. If the rise time is close to 1ns, then the peak voltage will be reduced to

$$E_{refl_{\max}} \sim 56mV$$

This is much less than 0.44V for $t_r = 80ps$ and a small amount of noise will introduce an error in estimation. Similarly, it can be shown that the voltage waveform becomes far from an ideal exponential decay when the rise-time is increased.

5.4 Estimation of *L* based on the area (Transmission case)

Unlike a small capacitance, the small inductance measurement is much more difficult. The time-domain approach shown here can be used. In many cases it is difficult to find the accurate exp(-1) time or slope *m* from the measured data due to a slow rise-time. Also these techniques are sensitive to measurement errors. In the following section, we will show the area under the reflected voltage can be used for estimating L.

The right figure shows the voltage observed at the unknown inductor. If there is no resistor, the final voltage is 0 (or the final current is ΔV /Rs). The initial current I(t=0) = 0. We can express the area under the voltage waveform is

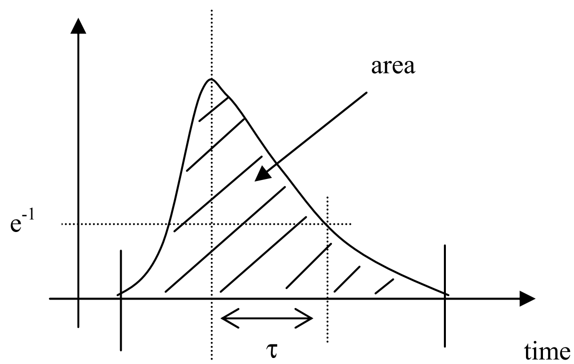
$$V_{inductor} = L\frac{dI_{inductor}}{dt}$$

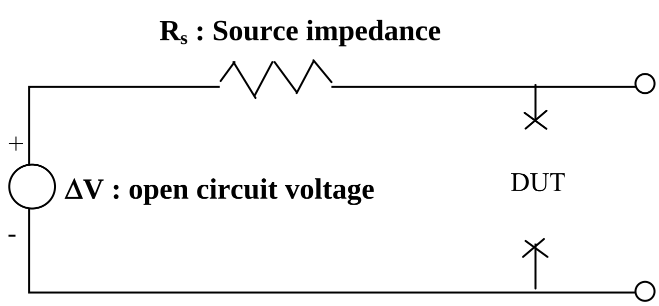
$$\int_0^\infty V_{inductor}(t)dt = L\int_0^\infty \frac{dI_{inductor}(t)}{dt}dt$$

$$\int_0^\infty V_{inductor}(t)dt = L[I(\infty) - I(0)]$$

$$\text{area} = L[I(\infty) - I(0)]$$

$$L = \frac{(area)}{\Delta I} = \frac{(area)(R_s)}{\Delta V}$$





R_s=equivalent source impedance due to source
ΔV is the open circuit voltage.

Example of Small Inductance Measurement.

The following figures show how to find the unknown inductance from the measured time-domain data. Unlike a small capacitor, measuring a small inductor with TDR is a challenging task. The time constant of series or parallel RL circuits has exp(-tL/R). If the value of L is small, R cannot to too large to observe a time-domain response within a reasonable time.

$$\tau = \frac{L}{R}$$

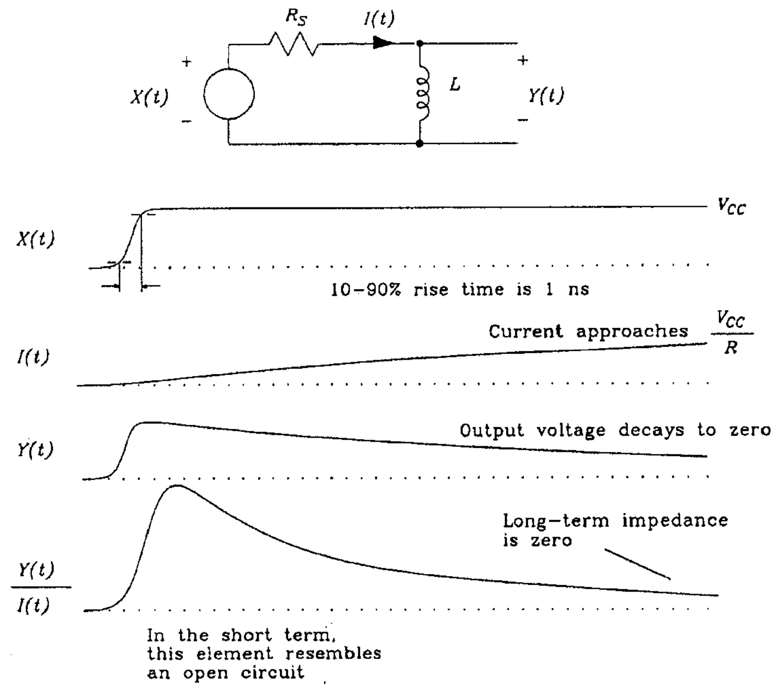


Figure 1.10 Instantaneous resistance of a perfect inductor.

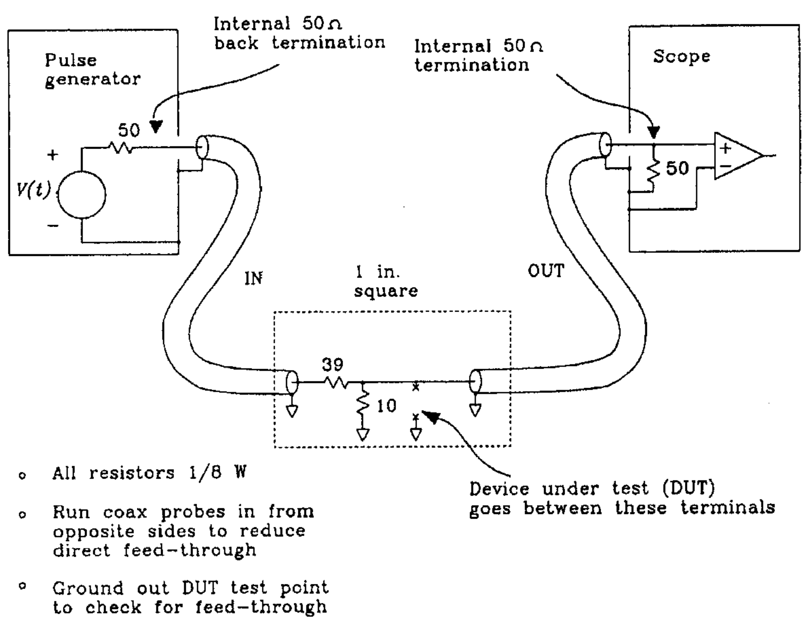
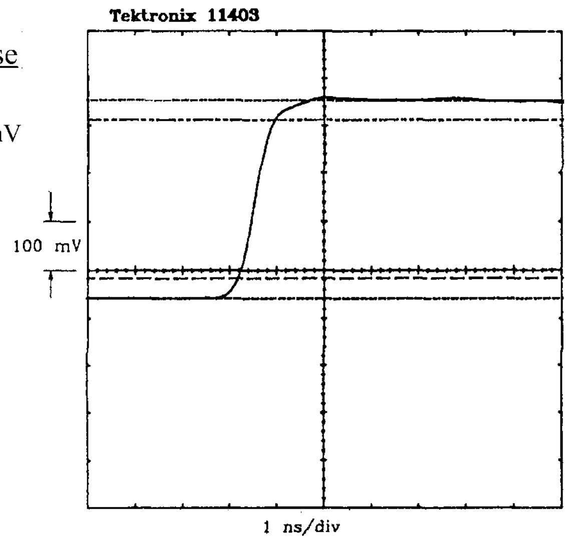


Figure 1.11 A 7.6-Ω lab setup for measuring inductance.

System response

Open circuit voltage ~ 418mV



Peak to peak, 417.6 mV	Rise, 788.1 ps			

Figure 1.12 *Open-circuit response of a 7.6-Ω inductance test setup.*

$89 // 10 // 50 = 7.6\Omega$

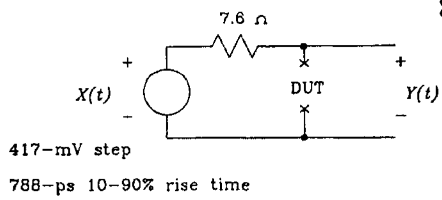


Figure 1.13 Thevenin equivalent of a 7.6-Ω inductance test setup.

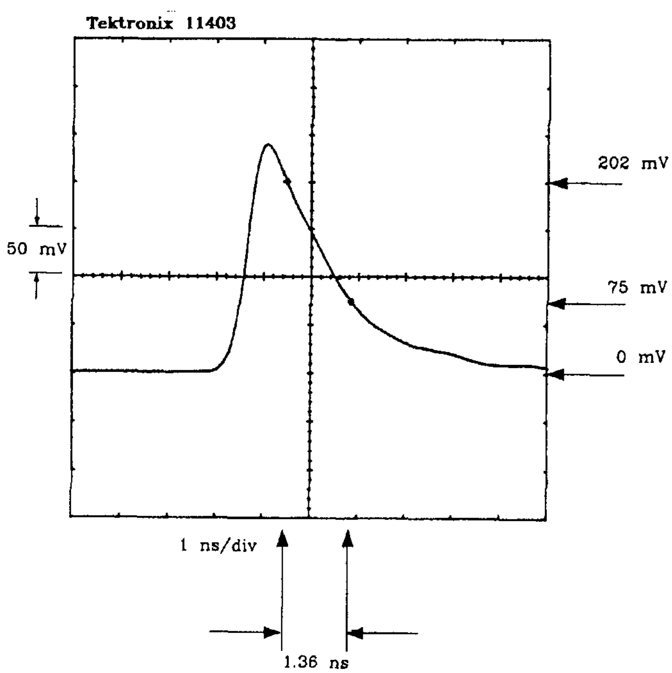


Figure 1.14 Decaying exponential response of a 7.6-Ω inductance test setup.

Based on the time constant

Find e^{-1} time

$$L = R\tau = (7.6)(1.36\times10^{-9}) = 10.3nH$$

R: must be small to get a large value of τ=L/R.

System response without *L*: 0.8ns (rise time of the system)
System response with *L*: 1.36ns (inductor decay time)

L estimation from the area

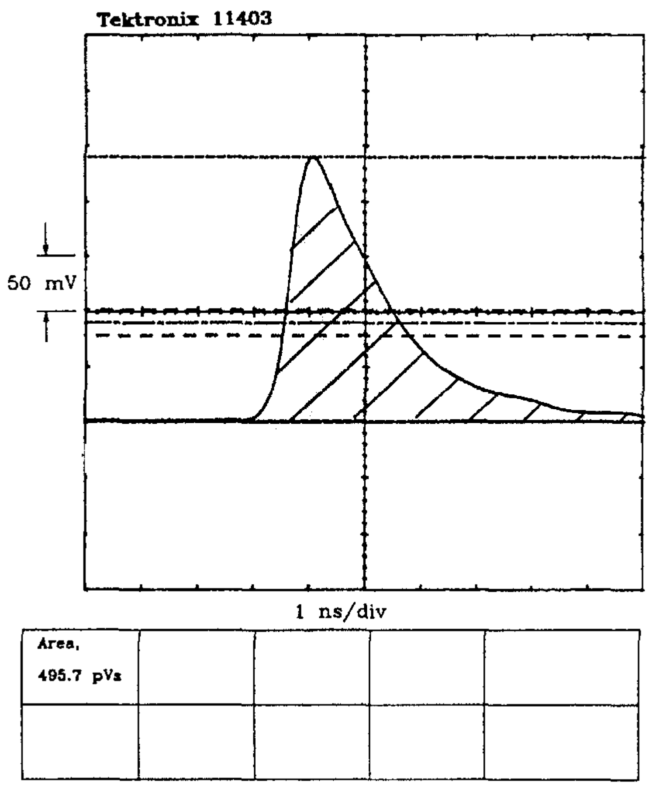


Figure 1.15 Area under the response of a 7.6-Ω inductance test setup.

$$\int_0^\infty V_{inductor}(t)dt = L\int_0^\infty \left(\frac{dI}{dt}\right)dt$$
$$= L[I(\infty) - I(0)]$$
$$= L\Delta I$$

$$\int_0^\infty V_{inductor}(t)dt = Area$$
$$L = \frac{Area}{\Delta I} = \frac{Area\cdot R_s}{\Delta V}$$

ΔV : open circuit voltage

$R_s = 7.6\Omega$

$$L = \frac{(495)(7.6)}{0.418} = 9nH$$

Rs=(50+39)//10//50=7.6 ohm

50+39:	toward source (pulse generator and series resistor)
10:	to ground
50:	toward scope (input impedance of the scope)

5.5 Small Capacitance Measurements

Unlike a small inductor, measuring a small capacitor with TDR is relatively easy. The time constant of series or parallel RC circuits has exp(-t/RC) which can be significantly increased by choosing a large R value.

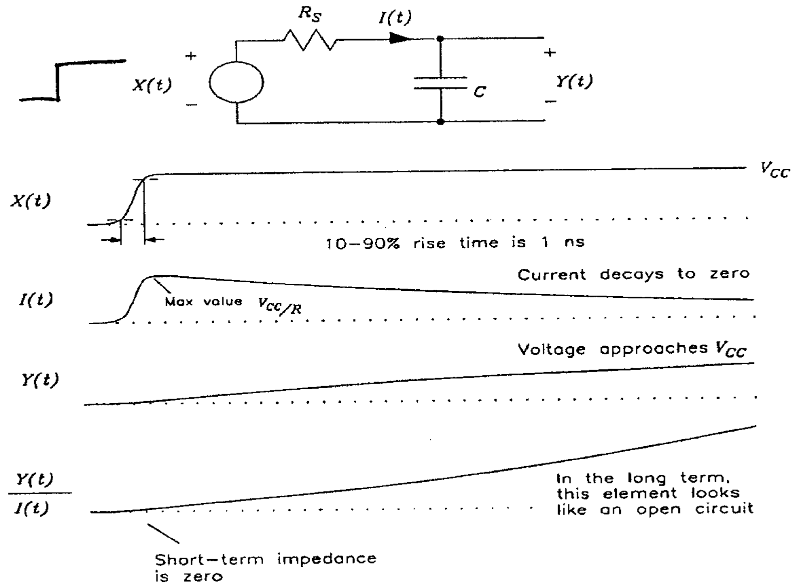


Figure 1.5 Step response of a perfect capacitor.

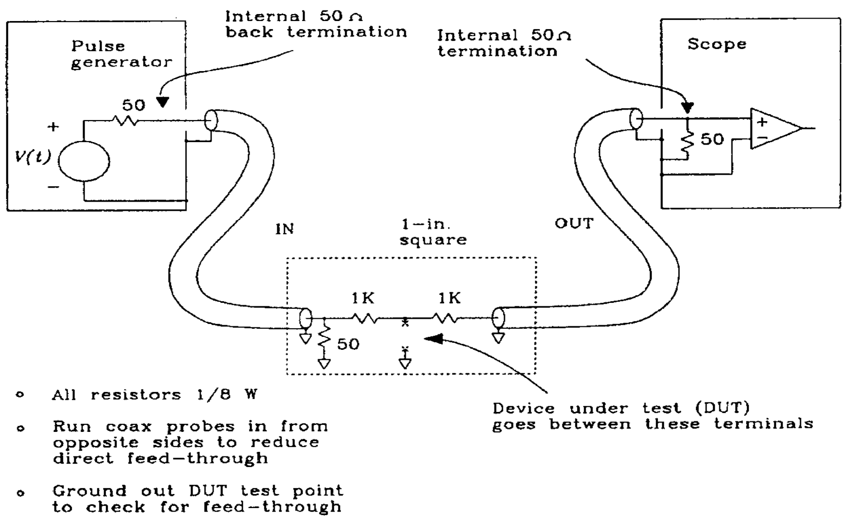
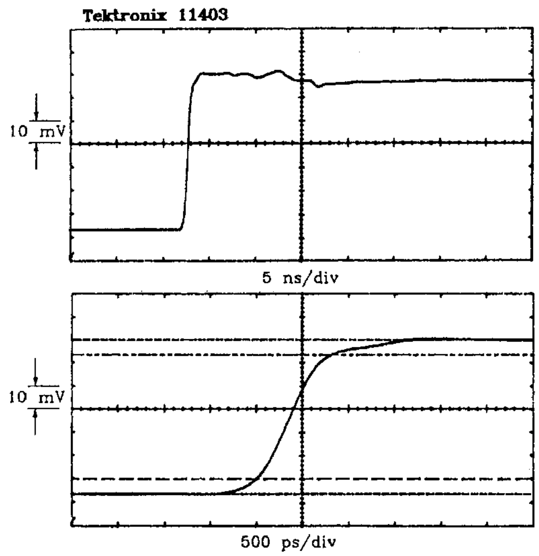


Figure 1.6 A 500-Ω lab setup for measuring capacitance.



Peak to peak, 67.16 mV	Rise, 818.9 ps			

Figure 1.7 Open-circuit response of a 500-Ω capacitance test setup.

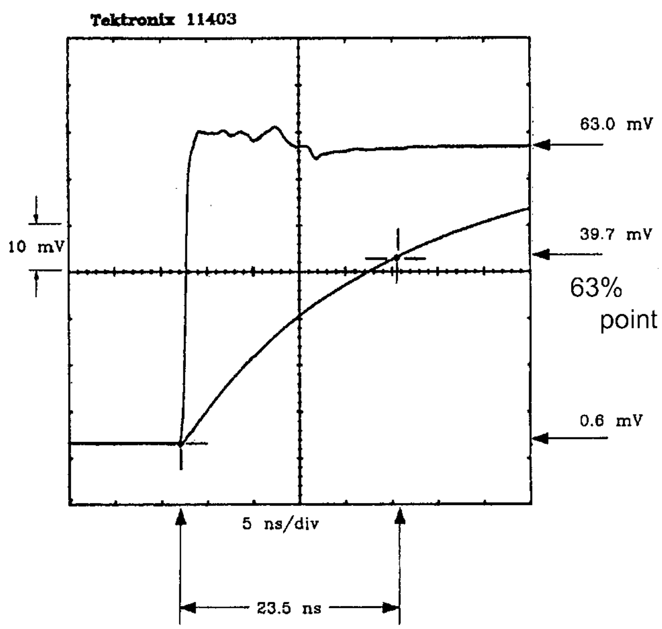
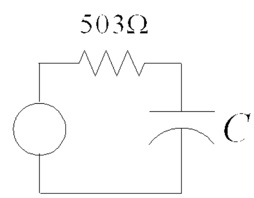


Figure 1.9 Finding a time constant using the 63% method.

$$1025 // 1050 = 503\Omega$$



$$C = \frac{23.5 * 10^{-9}}{503} = 467\,pF$$

Laplace Transform Theorem

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

	Name	Transform Pair
1.	Linearity	$ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$
2.	Scale change	$x(at) \leftrightarrow \frac{1}{a} X(\frac{s}{a}) \qquad a > 0$
3.	Time delay	$x(t - t_0) \leftrightarrow X(s) e^{-st_0} \qquad t_0 > 0$
4.	s-Shift	$e^{-at} x(t) \leftrightarrow X(s + a)$
5.	Multiplication by t^n	$t^n x(t) \leftrightarrow (-1)^n \frac{d^n X(s)}{ds^n} \qquad n = 1, 2, \ldots$
6.	Time differentiation	$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s) - \sum_{i=0}^{n-1} s^{n-1-i} x^{(i)}(0^-)$ where $x^{(i)}(t) = \frac{d^i x(t)}{dt^i}$
7.	Time integration	$y(t) = \int_{0^-}^{t} x(\lambda)\, d\lambda + y(0^-) \leftrightarrow \frac{X(s)}{s} + \frac{y(0^-)}{s}$
8.	Convolution	$x(t) * y(t) \leftrightarrow X(s) Y(s)$

The final two theorems present equal expressions rather than transform pairs. Also, these two theorems are valid only if the conditions stated in Chapter 7 are satisfied.

9.	Final value	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$
10.	Initial value	$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$

Table C.4 Laplace Transform Pairs

	$x(t)$	$X(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\dfrac{1}{s}$
3.	$\dfrac{t^n}{n!}u(t)$	$\dfrac{1}{s^{n+1}}$
4.	$e^{-at}u(t)$	$\dfrac{1}{s+a}$
5.	$\dfrac{t^n e^{-at}}{n!}u(t)$	$\dfrac{1}{(s+a)^{n+1}}$
6.	$\sin(\omega_0 t)\, u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$
7.	$\cos(\omega_0 t)\, u(t)$	$\dfrac{s}{s^2+\omega_0^2}$
8.	$t\sin(\omega_0 t)\, u(t)$	$\dfrac{2\omega_0 s}{\left(s^2+\omega_0^2\right)^2}$
9.	$t\cos(\omega_0 t)\, u(t)$	$\dfrac{\left(s^2-\omega_0^2\right)}{\left(s^2+\omega_0^2\right)^2}$
10.	$e^{-at}\sin(\omega_0 t)\, u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$
11.	$e^{-at}\cos(\omega_0 t)\, u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$
12.	$2Ce^{-at}\cos(\omega_0 t)\, u(t)$ $+\,2De^{-at}\sin(\omega_0 t)\, u(t)$	$(C+jD)\big/(s+a+j\omega_0)$ $+\,(C-jD)\big/(s+a-j\omega_0)$
13.	$Ee^{-at}\cos(\omega_0 t)\, u(t)$ $+\,\left((F-Ea)\big/\omega_0\right)e^{-at}\sin(\omega_0 t)u(t)$	$\dfrac{Es+F}{s^2+es+f}$ $\quad a = e/2$, $\ \omega_0=\sqrt{f-a^2}$