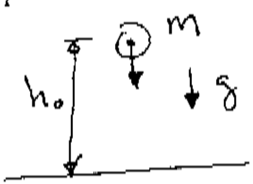
DIMENSIONAL ANALYSIS

Take the characteristic variables that the problem depends on and build a non-dimensional expression to account for that dependency.

Examples: (1)



What is the time it takes the stone to reach the ground?

$$t_f = f(m, g, h_o)$$

It does not depend on anything else.

Dimensions: $[h_o] = L$

$[g] = L T^{-2}$

$[m] = M$

$[t_f] = T$

We need to have a time on both sides of the equation:

$$\left[\sqrt{\frac{h}{g}}\right] = T$$

$$t_f = \sqrt{h_o/g} \cdot f(m)$$

$$\frac{t_f}{\sqrt{h_o/g}} = f(m)$$

: to be correct this expression needs to non-dimensional.

Since we cannot make m non dimensional (no combination of g, h, t_f contains M) the problem cannot depend on M.

$$\frac{t_f}{\sqrt{h_o/g}} = \text{constant}$$

We know that the solution is

$\frac{dV}{dt} = -g$; $\frac{dh}{dt} = V(t) \Rightarrow t_f = \sqrt{2 h_o/g}$ from the solution to the O.D.E

② Pressure loss in a pipe: $\Delta p = f(D, L, \rho, \mu, v)$

How about Q?

Q depends on $\frac{\pi D^2}{4}$ and v, so it is not an independent variable. We can use it instead of v (or D) but we can not use all three of them.

$$[\Delta P] = \frac{M L T^{-2}}{L^2} = M L^{-1} T^{-2}$$

$$\left.\begin{aligned} [D] &= L \\ [L] &= L \\ [\rho] &= M L^{-3} \\ [\mu] &= M L^{-1} T^{-1} \\ [v] &= L T^{-1} \end{aligned}\right\}$$

form a group that has dimensions of pressure: ρv^2

$$\frac{\Delta P}{\rho v^2} = f(D, L, \mu, \rho, v^2)$$

To complete this we need to form dimensionless groups with the variables we haven't used yet: D, L, μ

An easy one: $\frac{L}{D}$;

the other one $\mu / \rho v D$

$$\frac{\Delta P}{\rho v^2} = f\left(\frac{L}{D}, \frac{\mu}{\rho v D}\right)$$

If we know that the pressure loss is proportional to the length (conditions are uniform along the pipe in terms of geometry, rugosity, flow perturbations, etc.) then

$$\frac{\Delta P}{\rho v^2} = \frac{L}{D} f\left(\frac{\mu}{\rho v D}\right)$$

$$\boxed{\frac{\Delta P/L}{\rho v^2/D} = f\left(\mu/\rho v D\right)}$$

$\frac{\rho v D}{\mu}$ = Reynolds number, and is THE MOST IMPORTANT NON-DIMENSIONAL PARAMETER IN FLUID MECHANICS. For reasons completely different from this analysis of flow in pipes.

To carry out this analysis rigorously, we have Buckingham - Pi Theorem to guide us:

If an equation involving K variables is dimensionally homogeneous, then it can be reduced to a relationship between k-r dimensionless groups, where r is the minimum number of dimensions involved in the problem.

It is a constructive theorem:

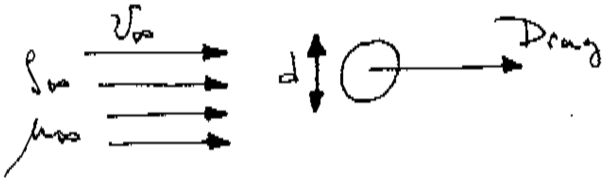
1. Identify the physical variables involved in the problem: k
2. Express each variable in term of its fundamental dimensions: MLTΘ system. r
3. Determine (k-r) pi terms are necessary to describe the problem.

4. Choose r variables to form the non-dimensional Π groups. They need to be chosen with care so that all dimensions are represented independently:

$$[\phi_1] = M^{a_1} L^{b_1} T^{c_1} \Theta^{d_1} \ldots$$
$$[\phi_2] = M^{a_2} L^{b_2} T^{c_2} \Theta^{d_2} \ldots \qquad \text{rank}\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & \ldots \\ \vdots & & & & \\ a_r & b_r & c_r & d_r & \cdots \end{pmatrix} = r$$
$$\vdots$$
$$[\phi_r] = M^{a_r} L^{b_r} T^{c_r} \Theta^{d_r} \ldots$$

5. Form $k-r$ non dimensional Π groups using the k variables chosen in 4 and everyone of the other $k-r$ variables

Example: Drag on a cylinder: $D = f(d, \rho_\infty, \mu_\infty, V_\infty)$



$K = 5$ ①

$[D] = MLT^{-2}$
$[d] = L$ $\qquad r = 3 \quad M, L, T$
$[\rho_\infty] = ML^{-3}$ $\qquad$ ②
$[\mu_\infty] = ML^{-1}T^{-1}$
$[V_\infty] = LT^{-1}$

③ $k - r = 2$ non-dimensional parameters.

④ Choose 3 variables: d, ρ_∞, V_∞ (Reynolds number)

$$\frac{D}{\rho_\infty V_\infty^2 d^2} = f\left(\frac{\mu_\infty}{\rho V_\infty d}\right) \Rightarrow \quad C_D = f(Re)$$

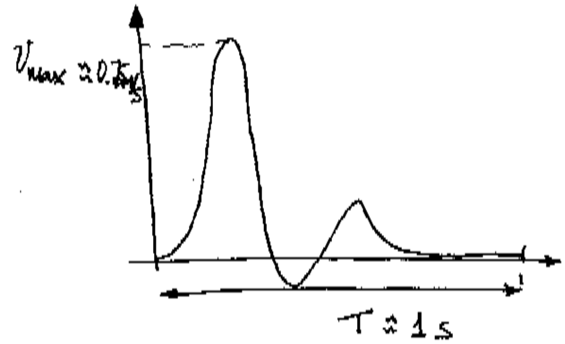
$$C_D = \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 d^2} \qquad Re = \frac{\rho_\infty V_\infty d}{\mu_\infty}$$

Crosssection Area of the object

Dynamic Similarity

If we want to test the behaviour of a phenomenon in a laboratory experiment, we need to match the values of the relevant physical parameters between the experiment and the real phenomenon.

Flow in the human Aorta: $V_{max} = 0.75\,m/s$



$f \simeq 1\,Hz$

$d \simeq 2.5 \cdot 10^{-2}\,m$ (1 in)

$\rho_{blood} \simeq 10^3\,kg/m^3$

$\mu_{blood} \simeq 2.7 \cdot 10^{-3}\,\frac{kg}{m \cdot s}$ @ 37°C

$4 \cdot 10^{-3}\,\frac{kg}{m \cdot s}$ @ 20°C

Experiment: $d = 10^{-1}\,m$ (4 in)

Characteristic non dimensional numbers are Re, St.

$$\tau_w = f(d, \mu, \rho, T, V_{max}) \qquad \left.\begin{matrix} k = 6 \\ r = 3 \end{matrix}\right\} \; k - r = 3$$

$$[\tau_w] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

$[d] = L$ ←

$[f] = T^{-1}$

$[V_{max}] = LT^{-1}$ ←

$[\rho] = ML^{-3}$ ←

$[\mu] = ML^{-1}T^{-1}$

$$\frac{\tau_w}{\rho V_{max}^2} = f\left(\frac{\mu}{\rho V_{max} d}, \frac{f \cdot d}{V_{max}}\right)$$

$$\text{if } \left.\frac{\mu}{\rho V_{max} d}\right|_{real} = \left.\frac{\mu}{\rho V_{max} d}\right|_{lab}$$

$$\text{and } \left.\frac{f \cdot d}{V_{max}}\right|_{real} = \left.\frac{f \cdot d}{V_{max}}\right|_{lab} \Rightarrow \left.\frac{\tau_w}{\rho V_{max}^2}\right|_{real} = \left.\frac{\tau_w}{\rho V_{max}^2}\right|_{lab}$$

$$d_{lab} = 4\, d_{real} \Rightarrow \left(\frac{f}{V_{max}}\right)_{lab} = \frac{1}{4}\left(\frac{f}{V_{max}}\right)_{real}$$

$$\left(\frac{\mu}{\rho V_{max}}\right)_{lab} = 4\left(\frac{\mu}{\rho V_{max}}\right)_{real}$$

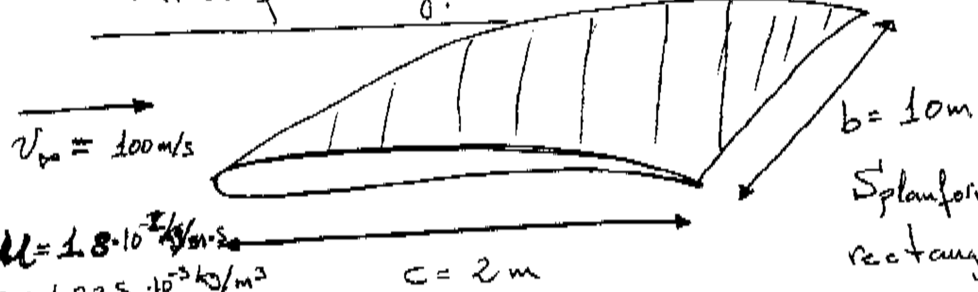
If we choose $f_{lab} = f_{real} \Rightarrow V_{max_{lab}} = 4 V_{max\,real}$ $\left(\simeq 3 \text{ m/s}\right)$

and $\left(\frac{\mu}{\rho}\right)_{lab} = 16\left(\frac{\mu}{\rho}\right)_{real}$ $\simeq \left(45 \cdot 10^{-3} \frac{kg}{m \cdot s}\right)$
(45 cP)
45 times the viscosity of water

$\underline{\tau_{w_{lab}} = 16\, \tau_{w_{real}}}$ → This could be good as τ_w is typically a very small quantity and therefore difficult to measure.

55

Aircraft wing.



$S_{planform} \simeq b \cdot c$ for a rectangular wing.

$\mu = 1.8 \cdot 10^{-5}\,kg/m \cdot s$

$\rho = 1.225 \cdot 10^{-3}\,kg/m^3$

$$\frac{L}{D} = f(\mu_{air}, V_\infty, \rho_{air}, c, b)$$

$$\frac{L}{\frac{1}{2}\rho_{air} V_\infty^2 \cdot c \cdot b} = f\left(\frac{\mu_{air}}{\rho_{air} \cdot V_\infty \cdot c}, \frac{b}{c}\right)$$

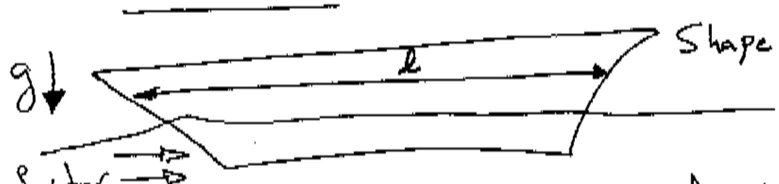
$$c_{lab} = 0.2\,m \rightarrow b_{lab} = 1.0\,m$$

$$\left(\frac{\mu_{air}}{\rho_{air} \cdot V_\infty \cdot c}\right)_{lab} = \left(\frac{\mu_{air}}{\rho_{air} \cdot V_\infty \cdot c}\right)_{real} \Rightarrow V_{\infty lab} = V_{\infty real} \cdot \underbrace{\frac{c_{real}}{c_{lab}}}_{= 10} \cdot \underbrace{\frac{(\mu_{air}/\rho_{air})_{lab}}{(\mu_{air}/\rho_{air})_{real}}}_{\simeq 1}$$

$$V_{\infty lab} \simeq 10 \cdot V_{real} \simeq 1000\,m/s.$$

!!!

Boat



$$D = f(\mu_w, \rho_w, V_\infty, l, g, \text{shape})$$

$$\frac{D}{\rho_w V_\infty^2 l^2} = f\left(\frac{\mu_w}{\rho_w V_\infty l}, \frac{g}{V_\infty^2/l}\right)$$

$$\boxed{\frac{V_\infty}{\sqrt{g \cdot l}} = F_r}$$

$$Re_{lab} = Re_{real} \; ; \; F_{r_{lab}} = F_{r_{real}}.$$

56

$$\left(\frac{V_\infty}{\sqrt{g\cdot l}}\right)_{lab} = \left(\frac{V_\infty}{\sqrt{g\, l}}\right)_{real}$$

if $g_{lab} = g_{real}$

and $\nu_{lab} = \nu_{real}$

$$\left(\frac{V_\infty \cdot l}{\nu}\right)_{lab} = \left(\frac{V_\infty \cdot l}{\nu}\right)_{lab}$$

$$\left(\frac{V_\infty}{\sqrt{l}}\right)_{lab} = \left(\frac{V_\infty}{\sqrt{l}}\right)_{real}$$

$$(V_\infty \cdot l)_{lab} = (V_\infty \cdot l)_{real}$$

You can satisfy one or the other but not both.

$l_{real} \simeq 100\ m$ $\qquad$ $\underline{\underline{l_{lab} \simeq 1\ m}}$

$V_{\infty\, real} \simeq 10\ m/s$ $\qquad$ $V_{\infty\, lab} = V_{\infty\, real} \cdot \sqrt{\dfrac{l_{lab}}{l_{real}}}$

$$V_{\infty\, lab} = 10\ m/s \cdot \frac{1}{10} \simeq \underline{\underline{1\ m/s}}$$

but then: $\dfrac{1\ m/s \cdot 1m}{\nu_{lab}} = \dfrac{10\ m/s \cdot 100m}{\nu_{real}} \Rightarrow \underline{\nu_{lab} \simeq \nu_{real}/1000}$

not feasible.

$$\underline{Re_{lab} \simeq 10^6,\ Re_{real} \simeq 10^9.}$$

Thanks to the asymptotic behaviour for large Reynolds numbers, the behaviour is reproduced in the lab to some extend.