Numerical Parameters for the Turbulent Wake

In class it was assumed that, in the 'far wake', the mean velocity deficit is of similarity form, i.e.,

$$\frac{\Delta U(x,y)}{U_s(x)} = f\left[\frac{y}{\ell(x)}\right], \tag{1}$$

where $\Delta U(x,y)$ is the mean velocity deficit, $U_s(x)$ is the local maximum of ΔU, $\ell(x)$ is a lateral length scale characterizing the flow, and $f(\eta)$ is a function describing the shape of the velocity deficit profile. For consistency with the similarity assumption, and with the conservation of the momentum flux $\mathcal{M}$, it was further shown that $U_s(x)$ and $\ell(x)$ behave in x as

$$\ell(x) = \mathcal{A}x^{1/2}\,, \text{ and } U_s(x) = \mathcal{B}x^{-1/2}\,, \tag{2}$$

where $\mathcal{A}$ and $\mathcal{B}$ are constants determined below. Finally, making a turbulent diffusivity assumption with constant turbulent viscosity, it was found that $f(\eta)$ satisfies

$$f'' + \alpha(f + \eta f') = 0\,, \tag{3}$$

where $\alpha = \dfrac{U_0\,\mathcal{A}}{2\,\mathcal{B}}R_T$, with U_0 the free stream speed, and R_T a turbulent Reynolds number defined by

$$R_T = \frac{U_s \ell}{\nu_T}\,, \tag{4}$$

with ν_T the (constant) turbulent viscosity. The exact solution of Equation (3) for f is

$$f(\eta) = f(0)\exp(-\frac{1}{2}\alpha\eta^2)\,. \tag{5}$$

Assuming the maximum of ΔU is at $y = 0$, then $f(0) = 1$.

In order to specify the length scale ℓ, the parameter α is chosen to be 1, i.e.,

$$\alpha = \frac{U_0\,\mathcal{A}}{2\,\mathcal{B}}R_T = 1\,. \tag{6}$$

Therefore, with $\eta = y/\ell$, when $y = \ell$, then $\eta = 1$ and

$$f(1) = \exp\left(-\frac{1}{2}\right) \approx 0.61\,, \tag{7}$$

that is, ℓ is defined as the y-value where ΔU has dropped to 0.61 of its peak value U_s.

Furthermore, using the definitions of the momentum flux $\mathcal{M}$ and the momentum thickness Θ, then

$$\mathcal{M} = \rho U_0^2\Theta = \rho\int_{-\infty}^{\infty} U_0(U_0 - U(x,y))\,dy = \rho U_0 U_s \ell(x)\int_{-\infty}^{\infty} f(\eta)\,d\eta = \rho U_0\mathcal{A}\mathcal{B}\sqrt{2\pi}\,, \tag{8}$$

using Equation (2) for U_s and ℓ, and since

$$\int_{-\infty}^{\infty} f(\eta)\,d\eta = \sqrt{2\pi}\,. \tag{9}$$

Therefore

$$U_0\Theta = \mathcal{A}\mathcal{B}\sqrt{2\pi}\,. \tag{10}$$

From Equation (6),

$$\mathcal{B} = \frac{1}{2}U_0\mathcal{A}R_T\,. \tag{11}$$

Plugging this into Equation (10) gives

$$U_0\Theta = \sqrt{2\pi}\frac{R_T}{2}\,U_0\,\mathcal{A}^2\,,\text{ or}$$

$$\mathcal{A} = \underbrace{\sqrt{\frac{2}{R_T}\frac{1}{\sqrt{2\pi}}}}_{C_1}\,\Theta^{1/2}\,. \tag{12}$$

Therefore, from Equation (2), $\ell(x) = C_1\Theta^{1/2}x^{1/2}$, or

$$\frac{\ell(x)}{\Theta} = C_1\left(\frac{x}{\Theta}\right)^{1/2}\,. \tag{13}$$

From laboratory data, it is found that the turbulent Reynolds number, which should be a constant for this problem since the product $U_s\ell$ is constant, and since ν_T has been assumed to be constant, has the approximate value

$$R_T \approx 12.5\,.$$

Therefore, from Equation (12),

$$C_1 \approx \left\{\frac{2}{12.5}\frac{1}{\sqrt{2\pi}}\right\} = 0.2526\,,\text{ and so}$$

$$\frac{\ell(x)}{\Theta} = 0.2526\left(\frac{x}{\Theta}\right)^{1/2}\,. \tag{14}$$

Again using Equation (6),

$$\mathcal{A} = \frac{2\mathcal{B}}{U_0R_T}\,; \tag{15}$$

plugging this into Equation (10) gives

$$U_0\Theta = \sqrt{2\pi}\frac{2\mathcal{B}}{U_0R_T}\mathcal{B} = \frac{\sqrt{2\pi}}{U_0}\frac{2}{R_T}\mathcal{B}^2\,,\text{ or}$$

$$\mathcal{B} = U_0\Theta^{1/2}\underbrace{\sqrt{\frac{R_T}{2\sqrt{2\pi}}}}_{C_2}\,. \tag{16}$$

With $C_2 = \sqrt{\dfrac{R_T}{2\sqrt{2\pi}}} = 1.579$, then

$$\frac{U_s(x)}{U_0} = 1.579\left(\frac{x}{\Theta}\right)^{-1/2}\,. \tag{17}$$

Solution to the Wake Equation

Consider the equation for $f(\eta)$, the similarity solution to the wake equation:

$$f'' + \alpha(f + \eta f') = 0\,,$$

where $(\cdot)'$ denotes differentiation with respect to η. Note that this can be written in the form

$$(f' + \alpha\eta f)' = 0\,.$$

Integrating this over η from 0 to η gives:

$$\int_0^\eta \frac{d}{d\eta}\left(\frac{df}{d\eta} + \alpha\eta f\right) d\eta = 0\,, \text{ or}$$

$$\left(\frac{df}{d\eta} + \alpha\eta f\right)\Big|_0^\eta = 0\,,$$

$$\frac{df}{d\eta} + \alpha\eta f - f'(0) = 0\,, \text{ so, finally}$$

$$\frac{df}{d\eta} + \alpha\eta f = f'(0)\,.$$

(Note that, due to the symmetry expected in the wake problem, it is expected that $f'(0) = 0$. But this will not be assumed here.) This last equation can be rewritten as

$$\left[f \exp(\frac{1}{2}\alpha\eta^2)\right]' \exp(-\frac{1}{2}\alpha\eta^2) = f'(0)\,, \text{ or}$$

$$\left[f \exp(\frac{1}{2}\alpha\eta^2)\right]' = f'(0)\exp(\frac{1}{2}\alpha\eta^2)\,.$$

Integrating this over η from 0 to η gives

$$f \exp(\frac{1}{2}\alpha\eta^2)\Big|_0^\eta = f'(0)\int_0^\eta \exp(\frac{1}{2}\alpha\eta'^2)\, d\eta'\,, \text{ or}$$

$$f \exp(\frac{1}{2}\alpha\eta^2) - f(0) = f'(0)\int_0^\eta \exp(\frac{1}{2}\alpha\eta'^2)\, d\eta'\,, \text{ so, finally}$$

$$f(\eta) = f(0)\exp(-\frac{1}{2}\alpha\eta^2) + f'(0)\exp(-\frac{1}{2}\alpha\eta^2)\int_0^\eta \exp(\frac{1}{2}\alpha\eta'^2)\, d\eta'\,.$$

The second term on the right-hand-side is zero in the cylindrical wake case, leaving

$$f(\eta) = f(0)\exp(-\frac{1}{2}\alpha\eta^2)\,.$$