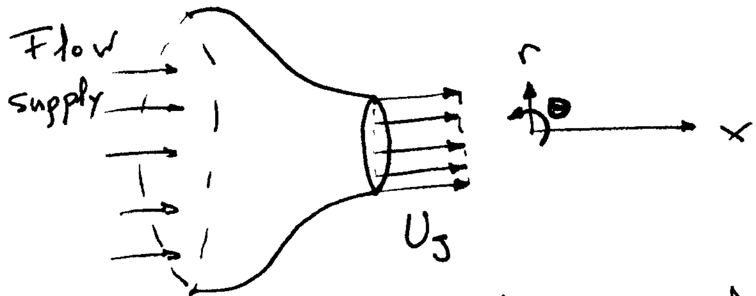
FREE SHEAR FLOWS

• ROUND JET



Axial velocity V_x is a function of x and r but not θ (axisymmetric).

We define the centerline velocity $U_0(x)$ as a scale for the velocity and the jet's half width $r_{1/2}(x)$ as a scale for the width.

Qualitatively, we see that $U_0(x)$ decays as x increases, while $r_{1/2}(x)$ grows as the jet spreads.

The equations of motion are:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r V_r\right) + \underbrace{\cancel{\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}}}_{0} + \frac{\partial V_x}{\partial x} = 0 \quad \text{(Continuity)}$$

$$\underbrace{\cancel{\frac{\partial V_r}{\partial t}}}_{0} + V_r\frac{\partial V_r}{\partial r} + \underbrace{\cancel{\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}}}_{0} - \underbrace{\cancel{\frac{V_\theta^2}{r}}}_{0} + V_x\frac{\partial V_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r} +$$

$$+\nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right] + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}}}_{0} - \underbrace{\cancel{\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}}}_{0} + \frac{\partial^2 V_r}{\partial x^2}\right\} \quad \text{(radial momentum)}$$

Axial momentum

$$\cancel{\frac{\partial V_x}{\partial t}}^{0} + V_r \frac{\partial V_x}{\partial r} + \cancel{\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}}^{0} + V_x \frac{\partial V_x}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} +$$

$$+ \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_x}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 V_x}{\partial \theta^2}}^{0} + \frac{\partial^2 V_x}{\partial x^2} \right]$$

By making the equations non-dimensional, we can simplify the problems by getting rid of negligible terms: $x \sim L$ $\quad V_x \sim U_0$ $\quad P \sim \rho U_0^2$

$r \sim \delta \quad V_r \sim ?$

- Continuity: $\frac{\overline{V_r}}{\delta} \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* V_r^*\right) + \frac{U_0}{L}\frac{\partial V_x^*}{\partial x^*} = 0$

therefore $\overline{V_r} \sim U_0 \frac{\delta}{L}$

- Axial momentum: $\frac{U_0^2 \cancel{\delta}}{L}\frac{1}{\cancel{\delta}} V_r^* \frac{\partial V_x^*}{\partial r^*} + \frac{U_0^2}{L} V_x^* \frac{\partial V_x^*}{\partial x^*} = -\frac{1}{\rho}\rho U_0^2 \frac{1}{L}\frac{\partial P^*}{\partial x^*}$

$$+ \nu \left[\frac{1}{\delta}\frac{U_0}{\delta} \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial V_x^*}{\partial r^*}\right) + \frac{U_0}{L^2}\frac{\partial^2 V_x^*}{\partial x^{*2}} \right]$$

order of magnitude of the terms is:

$$\frac{U_0^2}{L} \qquad \frac{U_0^2}{L} \qquad \frac{U_0^2}{L} \qquad \nu \frac{U_0}{\delta^2} \qquad \cancel{\nu \frac{U_0}{L^2}}$$

③

The second derivative along the ial direction is much smaller than in the radial direction, just as in the boundary layer equations.

δ, the thickness of the jet, is such that the viscous terms are of the same order as the convective ones: $\frac{U_0^2}{L} \sim \nu \frac{U_0}{\delta^2} \Rightarrow \left(\frac{\delta}{L}\right)^2 \sim \frac{\nu}{U_0 L} \left(\frac{1}{Re}\right)$

• <u>Radial momentum:</u> $\frac{U_0^2 \cancel{\delta}}{L^2} \frac{1}{\cancel{\delta}} v_r^* \frac{\partial v_r^*}{\partial r^*} + \frac{U_0^2 \delta}{L} \frac{1}{L} v_x^* \frac{\partial v_r^*}{\partial x^*} =$

$$= -\frac{1}{\rho} \rho U_0^2 \frac{1}{\delta} \frac{\partial p^*}{\partial r^*} + \nu \left\{ \frac{U_0 \delta}{L} \frac{1}{\delta^2} \frac{\partial}{\partial r^*}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* v_r^*\right)\right] + \frac{U_0 \delta}{L} \frac{1}{\delta^2} \frac{\partial^2 \cancel{v}}{\partial x^{*2}} \right\}$$

$$\underbrace{\frac{U_0^2 \delta}{L^2} \qquad \frac{U_0^2 \delta}{L^2}}_{\text{convective terms}} \qquad \underbrace{\frac{U_0^2}{\delta}}_{\text{pressure gradient}} \qquad \underbrace{\nu \frac{U_0}{L \cdot \delta} \qquad \nu \frac{U_0}{L \cdot \delta}}_{\text{viscous terms}}$$

$$\frac{\cancel{U_0^2}/\delta}{\frac{\cancel{U_0^2}\,\delta}{L^2}} \sim \left(\frac{L}{\delta}\right)^2 \sim Re \gg 1 \Rightarrow$$ Pressure gradient is dominant compared to inertia in the radial direction

$$\frac{U_0^{\cancel{2}}/\cancel{\delta}}{\nu \frac{\cancel{U_0}}{L \cdot \cancel{\delta}}} \sim \frac{U_0 \cdot L}{\nu} \sim Re \gg 1 \Rightarrow$$ pressure gradient is dominant compared to viscous terms.

The resulting equations are:

$$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial (r V_r)}{\partial r} = 0 \qquad \text{Continuity}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial r} \qquad \text{Radial momentum}$$

$$V_r \frac{\partial V_x}{\partial r} + V_x \frac{\partial V_x}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)$$

Axial momentum

These equations are identical to the boundary layer equations. If, as is true, for flow over a flat plate, there is no external pressure gradient imposed $P(r \to \infty, x) = P_\infty$ then, since there is no change in pressure with the radial direction, $\frac{\partial P}{\partial x} = 0$ everywhere in the flow. This is true for a jet expanding into an infinite medium which is stagnant. To first approximation the pressure is kept constant.

(5) We can manipulate the equations to obtain:

$$\rho r V_x\left[\frac{\partial V_x}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right]+\rho r\left(V_r\frac{\partial V_x}{\partial r}+V_x\frac{\partial V_x}{\partial x}\right)=$$

$$=\rho r\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) \Leftarrow \rho r V_x\,[\text{Eq. Continuity}]+\rho r\left[\text{Axial momentum}\right]$$

$$2\rho r\underbrace{V_x\frac{\partial V_x}{\partial x}}+\rho\underbrace{\left[V_x\frac{\partial}{\partial r}(rV_r)+rV_r\frac{\partial V_x}{\partial r}\right]}=\mu\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)$$

$$\cancel{2}\rho r\,\frac{1}{\cancel{2}}\frac{\partial V_x^2}{\partial x}+\rho\frac{\partial}{\partial r}\left(rV_rV_x\right)=\mu\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)$$

We can integrate this equation with respect to r to obtain:

$$\underbrace{\frac{d}{dx}\int_0^\infty 2\pi\rho\, r V_x^2\,dr}_{\substack{\text{Rate of change of the total flux of}\\ \text{axial momentum across a surface perpendicular}\\ \text{to the jet's axis as we move downstream from the nozzle}}}+\int_0^\infty 2\pi\rho\frac{\partial}{\partial r}\left(rV_rV_x\right)dr=\int_0^\infty \mu\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)2\pi dr$$

$$\frac{d}{dx}\int 2\pi\rho\, r V_x^2\, dr = -2\pi\rho\; r V_r V_x \Big|_0^\infty + 2\pi\mu\; r\frac{\partial V_x}{\partial r}\Big|_0^\infty$$

At $r=0 \Rightarrow V_r = 0$ and $\frac{\partial V_x}{\partial r} = 0$ because of symmetry

At $r\to\infty \Rightarrow V_r \to 0$ and $\frac{\partial V_x}{\partial r} \to 0$
$V_x \to 0$

and we'll hypothesize (later we'll confirm this hypothesis) that $V_r \cdot V_x \to 0$ and $\frac{\partial V_x}{\partial r} \to 0$ faster than $r\to\infty$.

$$\boxed{\frac{d}{dx}\int_0^\infty 2\pi\rho\, r V_x^2\, dr = 0}$$

Axial momentum is conserved !!! Flux is constant at any distance downstream.

This result is consistent with a similarity solution. Based on what we have learned about this flow, it makes sense to write $V_x = U_0(x)\, f(\xi)$ where $\xi = \frac{r}{r_{1/2}(x)}$

Continuity gives us: $\frac{dU_0}{dx}\cdot f(\xi) - \frac{\xi}{r_{1/2}}\frac{dr_{1/2}}{dx}U_0\cdot f'(\xi) +$

$$+\frac{1}{\xi r_{1/2}}\frac{1}{\cancel{r_{1/2}}}\frac{\partial}{\partial \xi}\left(\xi \cancel{r_{1/2}} V_r\right) = 0$$

$$v_r = \frac{1}{\xi} U_0 \frac{dr_{1/2}}{dx} \int_0^{\xi} \xi^2 f'(\xi)\, d\xi - r_{1/2} \frac{dU_0}{dx} \frac{1}{\xi} \int_0^{\xi} \xi f(\xi)\, d\xi$$

Substituting all this in the axial momentum equation we obtain:

$$\left[\frac{1}{\xi} U_0 \frac{dr_{1/2}}{dx} \int_0^{\xi} \xi^2 f'(\xi)\, d\xi - r_{1/2} \frac{dU_0}{dx} \frac{1}{\xi} \int_0^{\xi} \xi f(\xi)\, d\xi\right] \cancel{U_0} f'(\xi) \frac{1}{\cancel{r_{1/2}}} r_{1/2}^2 +$$

$$+ r_{1/2}^2 \cancel{U_0} f(\xi) \left[\frac{dU_0}{dx} f(\xi) + U_0 f'(\xi) \frac{-\xi}{r_{1/2}} \frac{dr_{1/2}}{dx}\right] = \frac{\nu}{\xi \cancel{r_{1/2}}} \cdot \frac{1}{\cancel{r_{1/2}}} \frac{\partial}{\partial \xi}\left(\frac{\cancel{r_{1/2}}}{\cancel{r_{1/2}}} \xi \frac{\partial \cancel{U_0} f}{\partial \xi}\right)$$

$$r_{1/2} U_0 \frac{dr_{1/2}}{dx} \frac{1}{\xi} \int_0^{\xi} \xi^2 f'(\xi)\, d\xi - r_{1/2}^2 \frac{dU_0}{dx} \frac{1}{\xi} \int_0^{\xi} \xi f(\xi)\, d\xi$$

$$r_{1/2}^2 \frac{dU_0}{dx} f^2(\xi) - r_{1/2} U_0 \frac{dr_{1/2}}{dx} \xi f'(\xi) = \nu \frac{1}{\xi} \frac{\partial}{\partial \xi}\left[\xi f'(\xi)\right]$$

For the similarity solution to exist, the equation has to be independent of x:

$$\left.\begin{aligned} U_0 r_{1/2} \frac{dr_{1/2}}{dx} &= \text{constant} \\ r_{1/2}^2 \frac{dU_0}{dx} &= \text{constant} \end{aligned}\right\} \quad \left.\begin{aligned} U_0 \frac{dr_{1/2}^2}{dx} \\ r_{1/2}^2 \frac{dU_0}{dx} \end{aligned}\right\} \frac{d}{dx}\left(U_0 r_{1/2}^2\right) = \text{constant}$$

$$U_0 \cdot r_{1/2}^2 = C_1 \cdot x$$

⑧

$$r_{1/2}^2 = \frac{c_1 x}{U_0(x)}$$

$$\frac{c_1 x}{U_0(x)} \frac{dU_0}{dx} = C_2$$

$$\frac{dU_0}{U_0} = \frac{C_2}{C_1}\frac{dx}{x} \Rightarrow \underline{\underline{U_0(x) = k \cdot x^{C_2/C_1}}}$$

To get the exact form of $U_0(x)$ and $r_{1/2}(x)$ we need to go back to the conservation of axial momentum:

$$\frac{d}{dx}\left(2\pi\rho \int_0^{\xi} \xi \, r_{1/2} U_0^2 f^2(\xi) \, r_{1/2} \, d\xi\right) = 0$$

$$2\pi\rho \underbrace{\frac{d}{dx}\left(U_0^2 r_{1/2}^2\right)}_{=0} \int_0^{\xi} \xi f^2(\xi) d\xi = 0$$

$$U_0^2 r_{1/2}^2 = C_3^2 \Rightarrow U_0 = \frac{C_3}{r_{1/2}}$$

$$\frac{C_3}{\cancel{r_{1/2}}} \cdot r_{1/2}^{\cancel{2}} = C_1 \cdot x$$

$$\boxed{r_{1/2} = \frac{C_1}{C_3} x} \Rightarrow \boxed{U_0 = \frac{C_3}{\frac{C_1}{C_3} x} = \frac{C_3^2}{C_1} \frac{1}{x}}$$

In conclussion, the spreading rate is constant $\frac{dr_{1/2}}{dx} = \text{constant}(S)$ and the evolution of the centerline velocity is inversely proportional to x.

Now that we know the scaling for $r_{1/2}(x)$ and $U_o(x)$, we can write the equations in non dimensional form and solve them. In order to avoid dealing with V_r in terms of the integrals of $f(\xi)$ we redo the formulation in terms of the stream function $\Psi(r,x)$ so that $V_x = \frac{1}{r}\frac{\partial \Psi}{\partial r}$ and $V_r = -\frac{1}{r}\frac{\partial \Psi}{\partial x}$

Since we defined $V_x = U_o(x) f(\xi)$, the relationship is $\Psi(r,x) = \int_0^{\xi} r\, U_o(x)\, f(\xi)\, dr$. We'll use a slightly different similarity variable: $\eta = \frac{r}{x} = S \cdot \xi$

so that $\Psi(\eta, x) = \int_0^{\eta} x\cdot\eta\, U_o(x) f(\eta)\, x\, d\eta$

$$\Psi(\eta, x) = \underbrace{x^2 U_o(x)}_{K\cdot x} \underbrace{\int_0^{\eta} \eta f(\eta)\, d\eta}_{F(\eta)}$$

given this definition: $V_x = \frac{1}{x\cdot\eta}\frac{1}{x} K\cdot x\, F'(\eta)$

$$V_r = -\frac{1}{x\cdot\eta}\left[K\cdot F(\eta) + K\cdot x\, F'(\eta)\left(\frac{-\eta}{x}\right)\right]$$

The axial momentum equation is then:

$$-\frac{k}{x}\left(\frac{F}{\eta} - F'\right)\left[\frac{k}{x^2}\left(\frac{F''}{\eta} - \frac{F'}{\eta^2}\right)\right] + \frac{k}{x}\frac{F'}{\eta}\left[\frac{-k}{x^2}\frac{F'}{\eta} + \frac{k}{x}\left(\frac{F''}{\eta} - \frac{F'}{\eta^2}\right)\left(\frac{-\eta}{x}\right)\right]$$

$$= \frac{\nu}{x\cdot\eta}\frac{1}{x}\frac{d}{d\eta}\left[x\cdot\eta\,\frac{k}{x^2}\left(\frac{F''}{\eta} - \frac{F'}{\eta^2}\right)\right]$$

$$\frac{k^2}{x^3}\left(-\frac{FF''}{\eta^2} + \frac{FF'}{\eta^3} + \cancel{\frac{F'F''}{\eta}} - \frac{F'^2}{\eta^2}\right) + \frac{k^2}{x^3}\cancel{\left(-\frac{F'^2}{\eta^2} - \frac{F'F''}{\eta} + \frac{F'^2}{\eta^2}\right)}$$

$$= \frac{\nu k}{x^3}\,\frac{1}{\eta}\frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right)$$

If we make $k = \nu$ then we simplify to an O.D.E.

$$\underbrace{\frac{FF'}{\eta^2} - \frac{FF''}{\eta} - \frac{F'^2}{\eta}}_{} = \frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right)$$

$$\frac{d}{d\eta}\left(-\frac{FF'}{\eta}\right) = \frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right)$$

Integrating once: $F'' - \frac{F'}{\eta} + \frac{FF'}{\eta} = \text{constant}$

Boundary conditions for this O.D.E. are:

$v_r = 0$ at $r=0 \Rightarrow F(\eta=0)=0$ (1) and $F'(\eta=0)=0$ (2)

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Because of symmetry $\frac{\partial v_x}{\partial r} = 0$ at $r = 0$ so

$$\left(\frac{F''}{\eta} - \frac{F'}{\eta^2}\right) \xrightarrow[\eta \to 0]{} 0 \quad \text{so} \quad F''(\eta = 0) = 0$$
(3)

at $\eta = 0$ $\quad F''(0) - \frac{F'(0)}{\eta} + \frac{F(0)\,F'(0)}{\eta} \to 0 = \text{Constant}$

The equation can then be rewritten as

$$\eta F'' - F' + FF' = 0 \implies \frac{d}{d\eta}\left(\eta F' - 2F + \frac{1}{2}F^2\right) = 0$$

and integrating again: $\eta F' - 2F + \frac{1}{2}F^2 = \text{Constant}\,(C_2)$

from the boundary condition we get that $C_2 = 0$

$$\eta \frac{dF}{d\eta} = 2F - \frac{1}{2}F^2$$

$$\int \frac{dF}{2F - \frac{1}{2}F^2} = \int \frac{d\eta}{\eta}$$

↓ to integrate this we need to write this in terms of simple fractions:

$$\frac{1}{2F\left(1 - \frac{1}{4}F\right)} = \frac{A}{2F} + \frac{B}{1 - \frac{1}{4}F} \implies \frac{A - \frac{1}{4}AF + 2BF}{2F\left(1 - \frac{1}{4}F\right)} = \frac{1}{2F\left(1 - \frac{1}{4}F\right)}$$

$$A = 1 \; ; \quad -\frac{1}{4}A + 2B = 0 \Rightarrow B = \frac{A}{8} = \frac{1}{8}$$

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$$\int \frac{dF}{2F} + \frac{1}{8}\int \frac{dF}{1-\frac{1}{4}F} = \int \frac{d\eta}{\eta}$$

$$\frac{1}{2}\ln F - \frac{1}{2}\ln(4-F) = \ln\eta + \text{Constant } (C_3)$$

$$\ln\left(\frac{F}{4-F}\right)^{1/2} = \ln\eta + c$$

$$\frac{F}{4-F} = a\eta^2 \quad \text{where} \quad a = e^{2c}$$

$$F(\eta) = \frac{4a\eta^2}{1+a\eta^2}$$

Based on this solution, we obtain:

$$V_x(x,r) = \frac{1}{r}\,\nu \times \frac{1}{x}\,F'(\eta) = \frac{8a\eta(1+a\eta^2) - 4a\eta^2 2a\eta}{(1+a\eta^2)^2}\frac{\nu}{r}$$

$$V_x(x,r) = \frac{8a\eta + \cancel{8a^2\eta^3} - \cancel{8a^2\eta^3}}{(1+a\eta^2)^2}\frac{\nu}{r} = \frac{8a\frac{\cancel{r}}{x}\cdot\frac{\nu}{\cancel{r}}}{\left[1+a\left(\frac{r}{x}\right)^2\right]^2}$$

$$U_x(x,r) = \underbrace{\frac{8a\nu}{x}}_{U_0(x)\sim\frac{1}{x}} \cdot \underbrace{\frac{1}{\left[1+a\left(\frac{r}{x}\right)^2\right]^2}}_{\text{self similar function}}$$

$$V_r(x,r) = \frac{4a\nu}{x}\,\frac{\eta(1-a\eta^2)}{(1+a\eta^2)^2}$$