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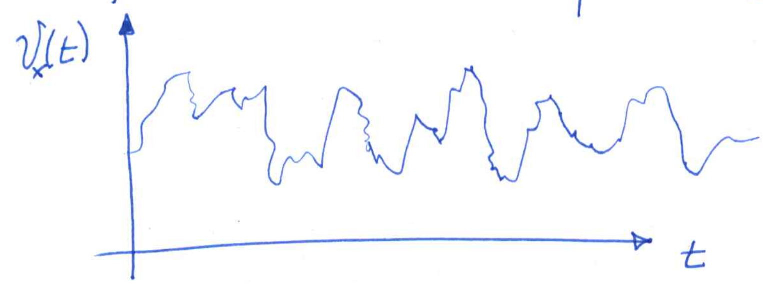
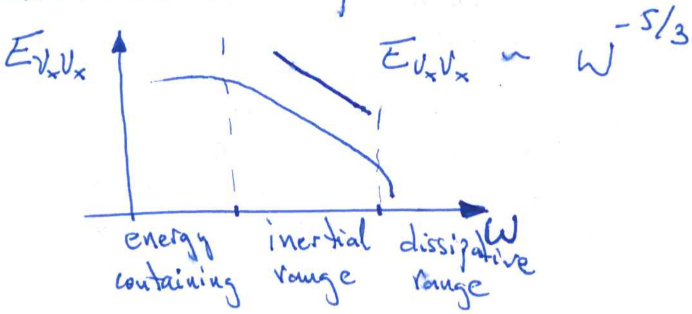
TURBULENCE

Characteristics of turbulence:

1. Spatial and temporal disorder. Randomness.



2. Universal scaling represent order in the midst of disorder. Spectrum.



3. Very effective transport of heat, mass and momentum (relative to solely molecular diffusivity).
   - Large scale structures: stirring
   - Small scale structures: increase area for molecular mixing

4. High Reynolds numbers. Non linearity. Chaos. Sensitive dependence to small changes in initial and boundary conditions

②

APPROACH:

1. Statistical theory of idealized (homogeneous and/or isotropic) turbulence.

2. Semiempirical/phenomenological theory of turbulence.

3. Coherent structures

4. Two point modeling $\overline{v_i(\vec{x},t)\, v_j(\vec{x}+\vec{r},\, t+\tau)}$

5. DNS: direct numerical simulation

6. LES: Large eddy simulation.

In all of them, the goal is to understand the dynamics of turbulent flows and to quantify the behaviour of statistical measures of relevant fluid mechanics variables.

Statistical tools:

$\vec{v}(\vec{x},t)$ is a random variable.

We are interested in the statistics of $\vec{v}(\vec{x},t)$

$\overline{v}$: average value

$\overline{v^2}$: Second moment (Variance)

$\overline{v^3}$: third moment (skewness)

$\overline{v^4}$: fourth moment (Kurtosis or flatness)

How do we define the statistical moments:

- Time averaging: $\overline{v} = \frac{1}{T}\int_0^T v(\vec{x}_0,t)\,dt$ where T is large compared to the fluctuation time but small compared to the characteristic time for large scale changes.

- Spatial averaging: $\overline{v} = \frac{1}{L}\int_0^L v(x,t_0)\,dx$
  Where L is large compared to the scale of fluctuations but small compared to the characteristic length scale of the flow.

$$= \frac{1}{L^2}\int_0^L\int_0^L v(x,y,t_0)\,dx\,dy$$

$$= \frac{1}{L^3}\int_0^L\int_0^L\int_0^L v(x,y,z,t_0)\,dx\,dy\,dz$$

- Ensemble averaging: $\overline{v} = \frac{1}{N}\sum_{i=1}^{N} v\left(\vec{x}_0, t+i\frac{2\pi}{\omega}\right)$
  where ω is the characteristic frequency of the process under study.

Reynolds Averaged Navier-Stokes

Reynolds proposed the following decomposition for the fluid variables: $v_i = \overline{v_i} + v_i'$

$$P = \overline{P} + P'$$

The incompressible Navier Stokes equation results in:

$$\frac{\partial(\overline{v_i}+v_i')}{\partial t} + (\overline{v_j}+v_j')\cdot\frac{\partial}{\partial x_j}(\overline{v_i}+v_i') = -\frac{1}{\rho}\frac{\partial(\overline{P}+P')}{\partial x_i} + \nu\frac{\partial^2(\overline{v_i}+v_i')}{\partial x_j\,\partial x_j}$$

(component in the i direction)

Averaging:

$$\overline{\frac{\partial(v_i+v_i')}{\partial t}} + \overline{(v_j+v_j')\frac{\partial}{\partial x_j}(\overline{v_i}+v_i')} = \overline{-\frac{1}{\rho}\frac{\partial(\overline{P}+P')}{\partial x_i}} + \overline{\nu\frac{\partial^2(\overline{v_i}+v_i')}{\partial x_j\,\partial x_j}}$$

Some important rules for averaging:

1. $\overline{v_i + v_i'} = \overline{\overline{v_i}} + \overline{v_i'}$: average of the sum is the sum of the averages.
2. $\overline{\frac{\partial v_i'}{\partial t}} = \frac{\partial}{\partial t}(\overline{v_i'})$: average of the derivative is the derivative of the average.
3. $\overline{v_i'} = 0$: the average of the fluctuating part of a variable is zero by definition.
4. $\overline{v_i' \cdot v_j'} \neq \overline{v_i'} \cdot \overline{v_j'}$: the average of the product IS NOT the product of the averages.

(5)

$$\frac{\partial}{\partial t}\overline{V_i} + \underbrace{\overline{V_j}\cdot\frac{\partial \overline{V_i}}{\partial x_j} + \overline{v_j'\frac{\partial v_i'}{\partial x_j}}}_{\text{non-linear term}} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{P} + \nu\frac{\partial^2 \overline{V_i}}{\partial x_j \partial x_j}$$

linear terms

Rewrite as Navier-Stokes for the average quantities:

$$\frac{\partial}{\partial t}\overline{V_i} + \overline{V_j}\frac{\partial}{\partial x_j}\overline{V_i} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{V_i}}{\partial x_j \partial x_j} \underbrace{- \overline{V_j'\frac{\partial V_i'}{\partial x_j}}}_{\text{This term depends on the fluctuations, not on the average quantities}}$$

Closure problem

We need to model $\overline{V_j'\frac{\partial V_i'}{\partial x_j}}$ or obtain an equation that predicts its behaviour. If we do that we will see that it depends on term of the form $\overline{V_i' V_j' V_K'}$ or $\overline{V_i' V_j' P'}$ $\Rightarrow$ a new closure problem.

Using continuity we can write $\overline{V_j'\frac{\partial V_i'}{\partial x_j}}$ as

$$\frac{\partial}{\partial x_j}\left(\overline{V_i' V_j'}\right) = \underbrace{\cancel{\overline{V_i'\frac{\partial V_j'}{\partial x_j}}}}_{0} + \overline{V_j'\frac{\partial V_i'}{\partial x_j}}$$

because $\frac{\partial}{\partial x_j}\left(\overline{V_j} + V_j'\right) = 0 \Rightarrow \begin{cases} \dfrac{\partial \overline{V_j}}{\partial x_j} = 0 \\[2ex] \dfrac{\partial V_j'}{\partial x_j} = 0 \end{cases}$

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$$\frac{\partial}{\partial t}\overline{V_i} + \overline{V_j}\frac{\partial \overline{V_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{V_i}}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{V_i' V_j'}\right)$$

Reynolds stress tensor

The Reynolds stress tensor represents flux of momentum that is not accounted for by the average velocity.

It is symmetric $\left(\overline{V_i' V_j'} = \overline{V_j' V_i'}\right)$ and has an isotropic and an anisotropic part:

$k = \frac{1}{2}\langle V_i' V_i' \rangle = \frac{1}{2}\left(\overline{V_1'^2} + \overline{V_2'^2} + \overline{V_3'^2}\right)$: turbulent kinetic energy

$$\overline{V_i' V_j'}^{D} = R_{ij}^{D} = R_{ij} - \frac{2}{3} k\, \delta_{ij}$$

Only the anisotropic part of R_{ij} affects the average velocity $\overline{V_i}$: Corrsin-Kistler equation

$$\frac{\partial\left(\overline{V_i' V_j'}\right)}{\partial x_j} = \frac{\partial R_{ij}^{D}}{\partial x_j} \underbrace{- \frac{2}{3}\frac{\partial k}{\partial x_j}\delta_{ij}}_{-\frac{2}{3}\frac{\partial k}{\partial x_i}}$$

$\Rightarrow$ this term can be "lumped" together with $-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i}$ so it has no effect

Modified pressure:

$$\mathcal{P} = \overline{P} + \frac{2}{3}\rho k \Rightarrow -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} - \frac{2}{3}\frac{\partial k}{\partial x_i}$$

Effect of vorticity on turbulent flows

Corrsin - Kistler equation:

$$v_i \boxed{\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = 0}$$ Irrotational flow

$$\overline{v_i' \frac{\partial v_i'}{\partial x_j}} - \overline{v_i' \frac{\partial v_j'}{\partial x_i}} = 0$$

$$\frac{\partial \left(\overline{v_i' v_i'}\right)}{\partial x_j} = \frac{\partial \left(\overline{v_i' v_j'}\right)}{\partial x_i} = 2 \frac{\partial k}{\partial x_j}$$

The divergence of the Reynolds stresses is equal to the gradient of the turbulent kinetic energy, which we just saw can be absorbed in the pressure term

$$\frac{\partial \left(\overline{v_i' v_j'}\right)}{\partial x_j} = 2 \frac{\partial k}{\partial x_i}$$

$$P = p + 2\rho k$$

There can be no turbulence without vorticity.
But the opposite is not true: vorticity in laminar flows

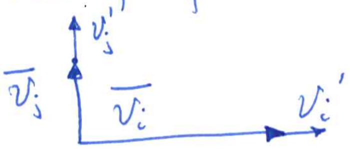
$$\frac{\partial \overline{v_i}}{\partial t} + \overline{v_j} \frac{\partial \overline{v_i}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 \overline{v_i}}{\partial x_j \partial x_j}$$

The velocity field responds to the exact same equation as a non-turbulent field with a pressure $P = p + 2\rho k$

$$\frac{\partial \overline{V_i}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{V_i}\,\overline{V_j}\right) = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{V_i}}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{v_i' v_j'}\right)$$

This has the form of a conservation of momentum for the average velocity $\overline{V_i}$, in terms of the average pressure $(\overline{P})$ and a new term $\overline{v_i' v_j'}$.

This term is a flux of $\overline{V_i}$ momentum due to the presence of the fluctuations in both the i direction (v_i') and in the perpendicular directions j (v_j').



The momentum flux in the i direction is due to

$$\underbrace{\overline{V_j}\,\overline{V_i}}_{1} + \underbrace{\cancelto{0}{\overline{v_j'\,\overline{V_i}}}}_{2} + \underbrace{\cancelto{0}{\overline{\overline{V_j}\,v_i'}}}_{3} + \underbrace{\overline{v_i' v_j'}}_{4}$$

1. The mean velocity $\overline{V_j}$ transports the momentum $\rho\overline{V_i}$

2 and 3. The fluctuating velocity v_j' transports the mean momentum $\rho\overline{V_i}$. But because v_j' has the same probability of being positive than negative (zero mean) the net result is zero.

The mean velocity $\overline{V_j}$ transports the fluctuating momentum $\rho v_i'$, but because v_i' has the same probability of being positive than negative (zero mean) the net result is zero.

4. The fluctuating velocity v_j' transports momentum in the i direction $\rho v_i'$. If v_j' and v_i' are uncorrelated, that is they have the same probability of being negative or positive regardless of the value of the other variable, then the net result would be zero. But in general

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it is not. We need a model for this correlation $\rho\,\overline{v_i' v_j'} \Rightarrow$ Reynolds stress tensor.

The simplest possible model is the definition of a turbulent viscosity. The resulting model is analog to the viscous stress being proportional to the deformation tensor: $\overline{v_i' v_j'} = -\nu_T \dfrac{\partial \overline{v_i}}{\partial x_j}$

The resulting equation is:

$$\frac{\partial \overline{v_i}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{v_i}\,\overline{v_j}\right) = -\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{v_i}}{\partial x_j \partial x_j} - \left(-\nu_T \frac{\partial^2 \overline{v_i}}{\partial x_j \partial x_j}\right)$$

We can add these terms together

$$\frac{\partial \overline{v_i}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{v_i}\,\overline{v_j}\right) = -\frac{\partial \overline{P}}{\partial x_j} + (\nu + \nu_T) \frac{\partial^2 \overline{v_i}}{\partial x_j \partial x_j}$$

Obviously, this model does not work for something as simple as Poiseuille flow: the turbulent profile is not a parabola, regardless of the value of the effective viscosity $\nu_{effec} = \nu + \nu_T$

In fact, it does not work well for any flow constrained by walls. But it does work reasonably well for free-shear flows: jets, wakes and mixing layers

Mean scalar equation

$$\frac{\partial \overline{\Phi}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{V_j}\,\overline{\Phi}\right) = \nu \frac{\partial^2 \overline{\Phi}}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{v_j' \phi'}\right)$$

The scalar flux on the rhs of the equation is the result of the additional transport due to fluctuations in the velocity and concentration of the scalar (and their correlation)

In parallel with the turbulent viscosity model, we can use the hypothesis that the turbulent scalar flux is related to a turbulent diffusivity D_t which is a function of the turbulent characteristics: $\overline{v_j' \phi'} = -D_T \frac{\partial \overline{\Phi}}{\partial x_j}$

In mass transport, the flux of a specie is typically considered proportional to the concentration gradient $\vec{f}_\alpha = -D_\alpha \nabla \phi_\alpha$: Fick's Law

For temperature: $\vec{q} = -K \nabla T$: Fourier's Law.

In Fick's Law, the characteristic length of the phenomenon (mass transfer) is orders of magnitude larger than the mean free path between collisions and averaging is natural (continuum assumption).

In turbulence, the size of the eddies is of the same order of magnitude as the characteristic lengthscale for the flow itself.

However, if there is a predominant direction for transport and a single velocity and length scales, it may be a good approximation (relative to its simplicity).

PRANDTL'S MIXING LENGTH THEORY.

In gas dynamics, one can show that

$\nu \sim a \cdot \lambda$ where λ is the mean free path between collisions and a is the wave speed (rms of the fluctuating molecular motion)

$\nu_T \sim v' l_m$ where l_m is the mixing length for the turbulent flow and v' is the rms of the fluctuating turbulent velocity (the dominant component)

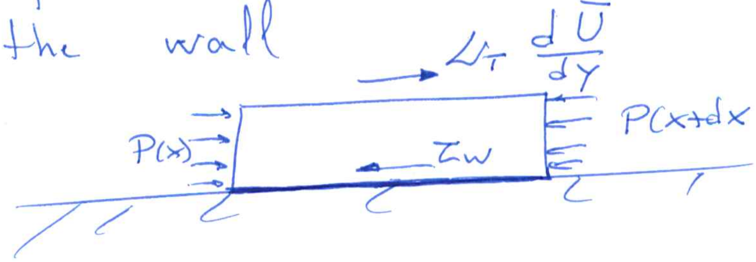
Prandtl used this to calculate the character of the turbulent boundary layer.

Near the wall $u' \sim u_\tau$: friction velocity at the wall $u_\tau = \sqrt{\frac{\tau}{\rho}}$

At the same time there is only one length scale, which is the distance to the wall y

so $\nu_T \sim u_\tau y$

If we take a small control volume close to the wall



$\frac{dP}{dx} = 0$ Zero pressure gradient boundary layer.

Then $\rho \nu_T \frac{d\bar{U}}{dy} = \tau w = \rho u_\tau^2$

$\nu_T \rightarrow K u_\tau y$

K: constant of proportionality, Von Karman constant

$$\rho K \cancel{u_\tau} y \frac{d\bar{U}}{dy} = \rho u_\tau^{\cancel{2}}$$

$$\frac{d\bar{U}}{dy} = \frac{u_\tau}{K y}$$

and integrating you get:

$$\boxed{\frac{\bar{U}}{u_\tau} = \frac{1}{K} \ln y + B}$$

logarithmic layer in the turbulent boundary layer

This is purely based on dimensional analysis, not physi

ENERGY CASCADE

Richardson introduced the idea of eddies:

"Big whorls have little whorls
which feed on their velocity
and little whorls have lesser whorls
and so on to viscosity
(in the molecular sense)."

Turbulence can be thought of as an assortment of eddies of different sizes. An eddy has characteristic length l and characteristic velocity $v(l)$. The largest eddies have size L, the characteristic length of flow (δ in a b.l or h the width of a channel) and velocity $v(l) = \sqrt{\overline{v'^2}}$. These large (energy containing) eddies break down into smaller eddies transferring their energy into smaller and smaller length scales. This process (energy cascade) goes on until the eddies are so small that the Reynolds number is of order one and the eddies no longer break up but rather are dissipated by viscosity. $Re_l = \frac{v(l) \cdot l}{\nu} \approx 1$

determines the smallest scale at which eddies exist AND defines this length scale as the one at which dissipation occurs.

KOLMOGOROV HYPOTHESIS

In 1941, Kolmogorov laid the foundation for the statistical treatment of turbulence.

Kolmogorov's 1^{st} hypothesis: Local isotropy

When the length scale used to study the turbulence is small enough, the turbulent motions are statistically isotropic. The memory of large scale forcing, boundary conditions, initial conditions, etc. is lost at small enough length scale.

Because of this lack of dependency of the small scales on the large scales features of the flow, Kolmogorov predicted that the small scale isotropic fluctuations would be "universal" for every turbulent flow. This state only depends on the rate of energy transfer from the large scales T_{EI} and the dissipation (which is related to the fluid viscosity ν).

In equilibrium, the rate of energy transfer from the large to the small scales is equal to the rate of dissipation: $\mathcal{T}_{EI} \simeq \epsilon$

This leads to KOLMOGOROV'S 1ST SIMILARITY HYPOTHESIS
The small scales $\ell < \ell_{EI}$ have a universal (self similar) state which is a function of ϵ and ν only.

Based on this, dimensional analysis tells us that there is only one length, time and velocity scales:

$\eta = \left(\nu^3/\epsilon\right)^{1/4}$: Kolmogorov lengthscale

$v_\eta = \left(\epsilon \cdot \nu\right)^{1/4}$: Kolmogorov velocity scale

$\tau_\eta = \left(\nu/\epsilon\right)^{1/2}$: Kolmogorov time scale.

$$Re_\eta = \frac{v_\eta \cdot \eta}{\nu} = 1$$

$\epsilon = \nu \left(\frac{v_\eta}{\eta}\right)^2$: rate of dissipation of turbulent kinetic energy is equal to the viscous dissipation at the smallest scales.

On the smallest scales, all high-Reynolds numbers velocity fields are ~~st~~statistically identical (when non-dimens sionalized with Kolmogo

The rate of dissipation of turbulent kinetic energy is given by the energy on the large, energy-containing scales that is transfered down to the dissipation scales: $\epsilon = \dfrac{v'^3}{L}$

The equality $\epsilon = \nu\left(\dfrac{v_\eta}{\eta}\right)^2$ and $\epsilon = \dfrac{v'^3}{L}$ sets the ratio (separation) between the large and the small scales in a turbulent flow:

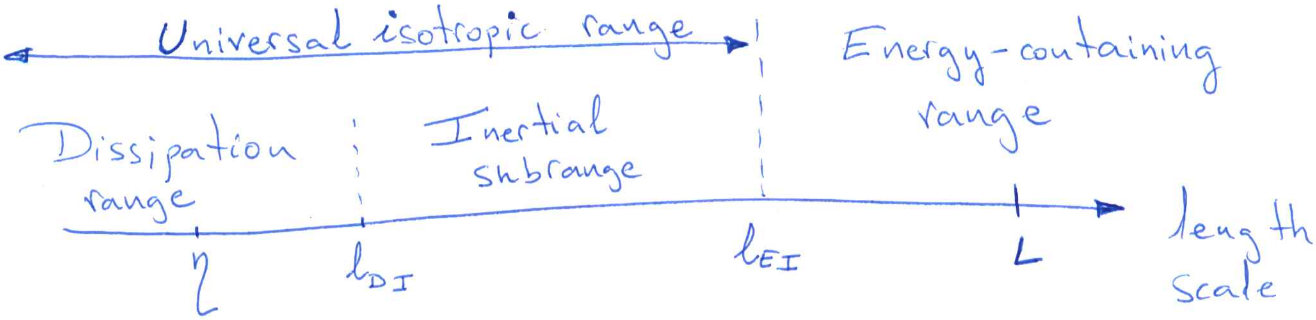
$$\nu\left(\frac{v_\eta}{\eta}\right)^2 = \frac{v'^3}{L} \Rightarrow \frac{\eta}{L} \sim \frac{\nu\,(\epsilon\cdot\nu)^{1/2}}{v'^3\,(\nu^3/\epsilon)^{1/4}} \sim \frac{(\epsilon\nu)^{3/4}}{v'^3}$$

$$\frac{\eta}{L} \sim \frac{v'^{9/4}\cdot\nu^{3/4}}{L^{3/4}\cdot v'^3} = Re_L^{-3/4}$$

Similarly: $\dfrac{v_\eta}{v'} \sim \dfrac{(\epsilon\,\nu)^{1/4}}{v'} \sim \dfrac{v'^{3/4}\nu^{1/4}}{L^{1/4}v'} \sim \left(\dfrac{\nu}{Lv'}\right)^{1/4} \sim Re_L^{-1/4}$

and $\dfrac{\tau_\eta}{L/v'} \sim \dfrac{(\nu/\epsilon)^{1/2}}{L/v'} \sim \dfrac{\nu^{1/2}L^{1/2}/v'^{3/2}}{L/v'} \sim \dfrac{\nu^{1/2}}{L^{1/2}v'^{1/2}} \sim Re_L^{-1/2}$

What this scaling shows is that, as the Reynolds number goes up, the ratio between the large and the small scales grow. $\dfrac{L}{\eta} \sim Re_L^{3/4}$. Therefore, we can define an intermediate range of scales l that are much larger than the dissipative scales η but much smaller than the energy-containing scales L.



This leads to Kolmogorov's 2ND Similarity Hypothesis: At sufficiently high Reynolds numbers, there is a range of scales $\eta \ll l \ll L$ such that the dynamics of the flow in these scales is only controlled by inertia (not by viscosity) and therefore is only determined by ϵ, and is universal in this scaling.

In the inertial range the scaling parameters are ϵ and the length l:

$$\mathcal{V}(l) = (\epsilon l)^{1/3} \sim v' \left(\frac{l}{L}\right)^{1/3}$$

$$\mathcal{T}(l) = \left(l^2/\epsilon\right)^{1/3} \sim T \left(\frac{l}{L}\right)^{2/3}$$

As we look at smaller length of the eddies within the inertial range $l \downarrow\downarrow$, the velocity and timescales decrease $\mathcal{V}(l) \downarrow\downarrow$ $\mathcal{T}(l) \downarrow\downarrow$

Energy Spectrum

The correlation between velocity fluctuations at two different points $\vec{x}$ and $\vec{x}+\vec{r}$ is typically called the 2point correlation $R_{ij}(\vec{r},\vec{x},t) = \overline{v_i'(\vec{x},t)\, v_j'(\vec{x}+\vec{r},t)}$

If we take the Fourier transform of this we get the velocity spectrum tensor $\Phi_{ij}(\vec{k},t)$

$$\Phi_{ij}(\vec{k},t) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}} R_{ij}(\vec{r},t)\, d\vec{r}$$

The Energy Spectrum is this same function integrated over 3 directions so that it looses direction information:

$$E(k,t) = \iiint_{-\infty}^{\infty} \frac{1}{2}\Phi_{ij}(\vec{k},t)\,\delta(|\vec{k}|-k)\, d\vec{k}$$

We go from a dependency on 3 wavenumber k_x, k_y, k_z to the dependency on only the magnitude of the wavenumber $\underline{\underline{k}}$

$$\int_{-\infty}^{\infty} E(k,t)\, dk = \frac{1}{2} R_{ii}(0,t) = \frac{1}{2}\overline{v_i'v_i'} = \text{Turbulent Kinetic energy}$$

The physical meaning of the Energy Spectrum is the amount of turbulent kinetic energy that is present in fluctuations (eddies) of a certain scale $\ell = \frac{2\pi}{k}$

According to Kolmogorov's 1ST similarity hypothesis the flow at the small scales $\ell < \ell_{EI}$ ($k > k_{EI}$) is universal and determined only by ν and ϵ.

Dimensional analysis then tells us that

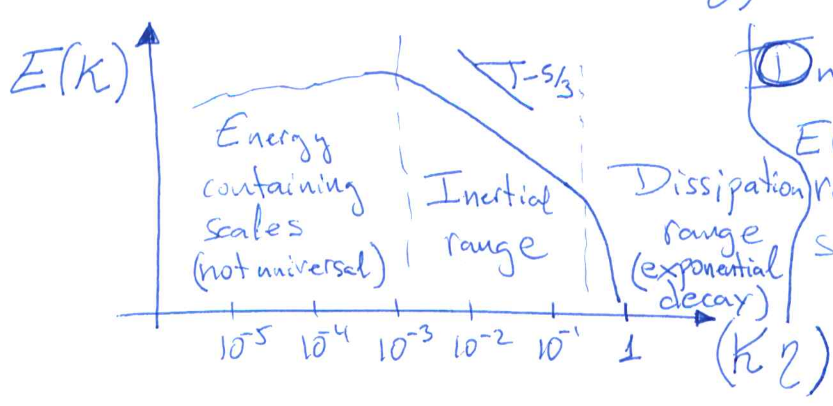
$$[E(k)] = \frac{(m/s)^2}{1/m} = \frac{m^3}{s^2}$$

$$[\nu] = m^2/s \qquad E(k) \sim (\epsilon \nu^5)^{1/4} \cdot \varphi(k\eta)$$

$$[\epsilon] = m^2/s^3 \qquad E(k) \sim v_\eta^2 \cdot \eta \cdot \varphi(k\eta)$$

According to Kolmogorov's 2ND similarity hypothesis the flow in the inertial range $\ell_{DI} < \ell < \ell_{EI}$ ($k_{DI} > k > k_{EI}$) is universal and is determined by ϵ alone.

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \cdot \Psi(k\eta)$$



① On a log-log plot $E(k)$ presents a region of constant $-\frac{5}{3}$ slope with respect to k ⇒ Inertial range