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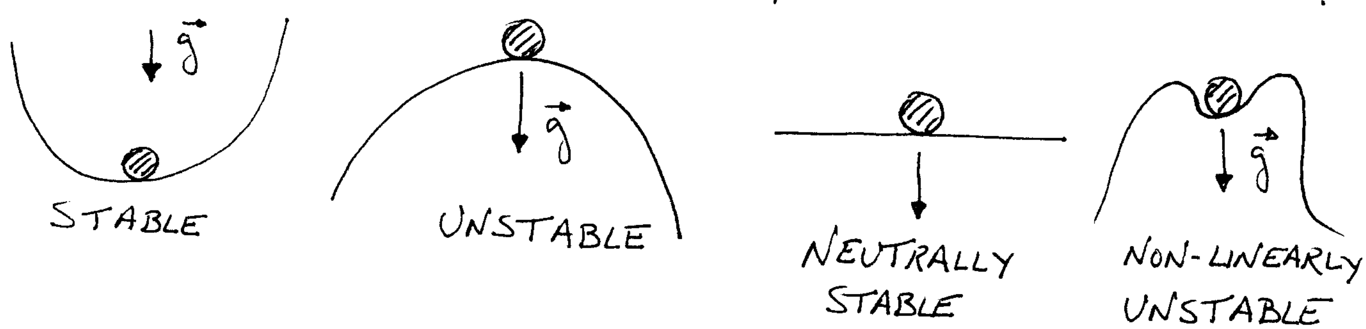
STABILITY

Solutions to the equations of motion don't necessarily appear in nature. Only STABLE solutions do.



We are going to focus on linear theory: infinitessimal small perturbations

NORMAL MODES

To analyze the stability to small perturbations we need to define a basic state and add the perturbations. For example, for Couette flow the basic flow is $\vec{V} = V_x(y)\,\vec{i}$ and the perturbation velocity is given by $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$. Similarly, the pressure is given by $\underbrace{P}_{\text{basic state}} + \underbrace{p}_{\text{perturbation}}$.

The normal mode method writes each perturbation as a superposition of normal Fourier modes: $v_x = v_x(y)\, e^{ikx + imz + \sigma t}$

Directions x and z are unbounded, whereas y is bounded. The coefficients of the resulting differential equation that determines the stability of the flow (e.g. Orr-Sommerfeld) depend on y but not on x, z or t. For solutions to be bounded the wavenumbers k and m need be real, but the frequency σ can be complex ($\sigma = \sigma_r + i\sigma_i$). Depending on the relationship between σ and k, m (called the dispersion relation) the problem will be stable if σ_r is negative for all k, m or it will be unstable to certain perturbation at wavelengths k, m for which σ_r first becomes positive.

If we plug in the fluid variables (basic state plus perturbation) into the Navier Stokes equation:

<u>x-momentum</u>

$$\frac{\partial (U+u)}{\partial t} + (U+u)\frac{\partial (U+u)}{\partial x} + v\frac{\partial (U+u)}{\partial y} + w\frac{\partial (U+u)}{\partial z} = -\frac{1}{\rho}\frac{\partial (P+p)}{\partial x} + \nu\nabla^2 (U+u)$$

We use $U(y)$, non-dimensionalize by U_0, L as characteristic velocity and length scales to get

$$\frac{\partial u}{\partial t} + (U+u)\frac{\partial u}{\partial x} + v\frac{\partial (U+u)}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial (P+p)}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 (U+u)}{\partial x^2} + \frac{\partial^2 (U+u)}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Now, we linearize the equation because the perturbations are small, so the product of two perturbation terms is negligible compared to terms with only one perturbation.

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \cancel{u\frac{\partial u}{\partial x}} + v\frac{\partial U}{\partial y} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{\partial P}{\partial x} - \frac{\partial p}{\partial x} +$$
$$+ \frac{1}{Re}\left(\cancel{\frac{\partial^2 U}{\partial x^2}} + \frac{\partial^2 U}{\partial y^2} + \cancel{\frac{\partial^2 U}{\partial z^2}} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The base state flow is a stand alone solution to the Navier-Stokes (the question we are trying to answer is whether it is a stable solution) so:

$$0 = -\frac{\partial P}{\partial x} + \frac{1}{Re}\frac{\partial^2 U}{\partial y^2}$$

When we substract this from the equation above:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u$$

and similarly we get from the other components of N-S and continuity:

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v$$
$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

We use a solution of the form.

$$\vec{v} = \vec{v}(y)\, e^{i(kx+mz-kct)}$$

$$p = p(y)\, e^{i(kx+mz-kct)}$$

as x and z are unbounded, k and m have to be real. We can consider only positive values. The wave speed c is complex $c = c_r + i c_i$ and the sign of c_r determines the sense of propagation for the perturbations.

$$(-ikc + Uik)\, u(y) + v\, U_y = -ik\, p(y) + \frac{1}{Re}\left[u_{yy} - (k^2+m^2)u\right]$$

$$(-ikc + Uik)\, v(y) = -p_y + \frac{1}{Re}\left[v_{yy} - (k^2+m^2)v\right]$$

$$(-ikc + Uik)\, w(y) = -im\, p(y) + \frac{1}{Re}\left[w_{yy} - (k^2+m^2)w\right]$$

$$ik\, u(y) + v_y + im\, w(y) = 0$$

SQUIRE'S THEOREM

To each unstable three dimensional perturbation, there is a corresponding two dimensional one which is more unstable (grows faster).

To prove this we transform the equation with:

$$\bar{k} = (k^2+m^2)^{1/2} \quad ; \quad \bar{c} = c \quad ; \quad \frac{\bar{p}}{\bar{k}} = \frac{p}{k}$$

$$\bar{k}\bar{u} = ku + mw \quad ; \quad \bar{v} = v \quad ; \quad \bar{k}\,\overline{Re} = k\,Re$$

$$\left.\begin{aligned} &iku(U-c) + vU_y = -ikp + \frac{1}{Re}\left[u_{yy} - (k^2+m^2)u\right] \\ \frac{m}{k} \times \Big\{ &ikw(U-c) = -imp + \frac{1}{Re}\left[w_{yy} - (k^2+m^2)w\right] \end{aligned}\right\} \text{adding them up:}$$

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$$i(ku+mW)(U-c) + vU_y = -i\left(kp+\frac{m^2}{k}p\right) + \frac{k}{\bar{k}\overline{Re}}\left(u_{yy} - \bar{k}^2 u\right) + \frac{m}{\bar{k}\overline{Re}}\left(W_{yy} - \bar{k}^2 W\right)$$

$$i\bar{k}\bar{u}(U-c) + \bar{v}U_y = -i\frac{\bar{k}^{\cancel{2}}\bar{p}}{\cancel{\bar{k}}} + \frac{(ku_{yy}+mW_{yy}) - \bar{k}^2(ku+mW)}{\bar{k}\,\overline{Re}}$$

$$\boxed{\begin{aligned} &i\bar{k}(U-c)\bar{u} + \bar{v}U_y = -i\bar{k}\bar{p} + \frac{1}{\overline{Re}}\left(\bar{u}_{yy} - \bar{k}^2\bar{u}\right) \\ &i\bar{k}(U-c)\bar{v} = -\bar{p}_y + \frac{1}{\overline{Re}}\left(\bar{v}_{yy} - \bar{k}^2\bar{v}\right) \\ &i\bar{k}\bar{u} + \bar{v}_y = 0 \end{aligned}}$$

To every three dimensional fluctuation (u, v, w) corresponds a two dimensional one that has a lower Reynolds number $\overline{Re} = \frac{k}{\bar{k}}Re$ and $\frac{k}{\bar{k}} \leq 1$ so that it will become unstable earlier (lower Re) or at the same Re will grow faster: growth rates are $e^{\bar{k}\bar{c}_i t}$ vs. $e^{kc_i t}$

$\bar{k} > k$ and $\bar{c} = c$ so the two dimensional perturbation grows faster.

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Using Squire's theorem we can restrict ourselves to two dimensional pertarbations: $u(y)\, e^{ik(x-ct)}$

$v(y)\, e^{ik(x-ct)}$

$p(y)\, e^{ik(x-ct)}$

This two dimensionality of the problem allows us to use the streamfunction: $u = \dfrac{\partial \Psi}{\partial y}$; $v = -\dfrac{\partial \Psi}{\partial x}$

$$\Psi = \psi\, e^{ik(x-ct)}$$

$u(y) = \psi_y$

$v(y) = -ik\psi$

and plugging these into the equations

$$ik(U-c)\,\psi_y + (-ik\psi)U_y = -ikp + \frac{1}{Re}\left(\psi_{yyy} - k^2\psi_y\right)$$

$$ik(U-c)(-ik\psi) = -p_y + \frac{1}{Re}\left(-ik\psi_{yy} + ik^3\psi\right)$$

To eliminate the pressure from this system of two equations and two unknowns, we take the y derivative of the first equation and add it up to the second equation multiplied by $(-ik)$:

$$ik(U-c)\,\psi_{yy} \cancel{+ikU_y\psi_y} \cancel{-ik\psi_y U_y} - ik\psi U_{yy} = \cancel{-ikp_y} + \frac{1}{Re}\left(\psi_{yyyy} - k^2\psi_{yy}\right)$$

$$+\; -ik^3(U-c)\,\psi \quad = \cancel{ikp_y} + \frac{1}{Re}\left(-k^2\psi_{yy} + k^4\psi\right)$$

dividing by (ik)

$$\boxed{(U-c)\left(\psi_{yy} - k^2\psi\right) - \psi U_{yy} = \frac{1}{ikRe}\left(\psi_{yyyy} - 2k^2\psi_{yy} + k^4\psi\right)}$$

This is the Orr - Sommerfeld equation for the stability of uni-directional (quasi parallel) flow. It is similar to the vorticity equation because we have removed the pressure term by taking a combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

INVISCID PARALLEL FLOWS

Taking the limit $Re \to \infty$ we get:

$$(U-c)\left(\Psi_{yy} - k^2\Psi\right) - \Psi U_{yy} = 0$$

If the flow is bounded by 2 walls at y_1 and y_2 the boundary conditions are $\Psi(y_1) = \Psi(y_2) = 0$

We have a homogeneous problem with homogeneous boundary conditions, and there is a "free" parameter that the problem depends on (c). This is an eigenvalue problem. The only solution to this problem is the trivial one $\Psi = 0$ except for some values of $c(k)$ (eigenvalues) that make the problem have non zero solutions (eigenfunctions).