(8)

We can write Rayleigh's equation as:

$$\psi_{yy} - \left(\frac{U_{yy}}{U-c} + k^2 \right) \psi = 0$$

This gives us a necessary condition for instability

RAYLEIGH'S CRITERION

Let's assume that there is an unstable mode $(c_i > 0)$

$$\psi = \psi(y) e^{ik(x-ct)} = \psi(y) e^{ik(x-c_r t)} \underbrace{e^{+kc_i t}}_{\text{growth rate}}$$

this means $U-c$ is not equal to zero (U is real).

By multiplying Rayleigh's equation by ψ^* (complex conjugate) and integrating we get

$$\int_{y_1}^{y_2} \psi^* \psi_{yy}\, dy - \int_{y_1}^{y_2} \frac{U_{yy}}{U-c} \psi \psi^* dy - \int_{y_1}^{y_2} k^2 \psi \psi^* dy = 0$$

$$\underbrace{\cancel{\psi^* \psi_y \Big|_{y_1}^{y_2}}}_{0} - \int_{y_1}^{y_2} \underbrace{\psi_y^* \psi_y}_{|\psi_y|^2} dy - \int_{y_1}^{y_2} \frac{U_{yy}}{U-c} \underbrace{\psi \psi^*}_{|\psi|^2} dy - \int_{y_1} k^2 \underbrace{\psi \psi^*}_{|\psi|^2} = 0$$

$Re[\psi(y_1) = \psi(y_2)] = 0$

So the equation becomes

$$\underbrace{\int_{y_1}^{y_2} |\psi_y|^2 dy + \int_{y_1}^{y_2} k^2 |\psi|^2 dy}_{\text{these two terms are real and positive definite}} + \int_{y_1}^{y_2} \frac{U_{yy}}{U-c} |\psi|^2 dy = 0$$

The first two terms are real and positive definite. The second term has a real and an imaginary part:

$$\int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi|^2 (U-c)^* \, dy = \int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi|^2 (U-c_r) \, dy$$

$$+ i \int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi|^2 c_i \, dy$$

For the equation to be satisfied

$$\int_{Y_1}^{Y_2} |\Psi_Y|^2 + k^2 |\Psi|^2 \, dy + \int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi|^2 (U - c_r + i c_i) \, dy = 0$$

it must be that $\displaystyle\int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi^2| \, dy = 0$ and that can only be true if U_{YY} changes sign somewhere between Y_1 and Y_2 (inflection point).

<u>There is another Theorem</u> (kind of a corolary from the previous one) that ~~looks~~ looks at the real part of the equation:

$$\underbrace{\int_{Y_1}^{Y_2} |\Psi_Y|^2 + k^2 |\Psi|^2 \, dy}_{\text{always positive}} + \underbrace{\int_{Y_1}^{Y_2} \frac{U_{YY}(U-c_r)}{|U-c|^2} |\Psi|^2 \, dy}_{U_{YY}\,(U-c_r) \text{ must change sign somewhere}} = 0$$

but we know already that $\int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi|^2 dy = 0$

So we can substract "anything" times $\int_{Y_1}^{Y_2} \frac{U_{YY}}{|U-c|^2} |\Psi|^2 dy$ from the equation to find

$$\int_{Y_1}^{Y_2} \frac{U_{YY}\left(U - c_r - \text{"anything"}\right)}{|U-c|^2} |\Psi|^2 dy = -\int_{Y_1}^{Y_2} |\Psi_Y|^2 + k^2 |\Psi|^2 dy < 0$$

What do we want "anything" to be?, $U_I - c_r$ is convenient:

$$U_{YY}\left(U - \cancel{c_r} - U_I + \cancel{c_r}\right) = U_{YY}\left(U - U_I\right)$$ has to be negative somewhere in the flow. U_I is the velocity at the inflexion point.

U_{YY} has to change signs so what does that imply

$U_{YY} < 0$ when $U < U_I$ and then $U_{YY} > 0$ when $U > U_I \Rightarrow$ it does not work.

$U_{YY} > 0$ when $U < U_I$ and then $U_{YY} < 0$ when $U > U_I \Rightarrow$ it DOES work.

$$W_z = -\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = -U_Y$$

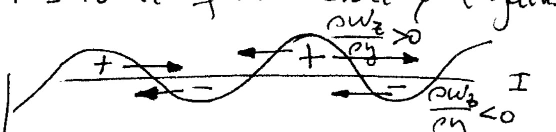
$$\frac{\partial W_z}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 V}{\partial y \partial x} = -U_{YY}$$

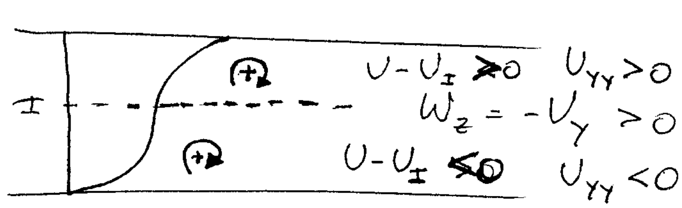
Fjørtøft's Theorem

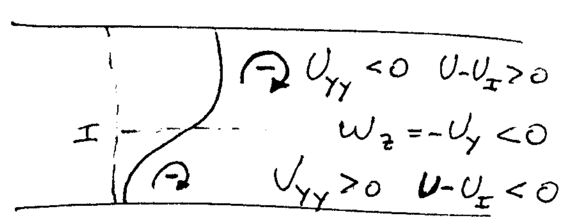
A necessary condition for the instability of an inviscid parallel flow is that $U_{YY}(U-U_I) < 0$ somewhere in the flow. This is a more restrictive condition that Rayleigh's criterion, but it is still only a necessary but not sufficient condition.

Both Rayleigh's and Fjortoft's Theorems can be interpreted ~~in~~ terms of vorticity:

U_{YY} is the rate of change of vorticity. If it changes sign it means that the vorticity magnitude goes through a maximum inside the flow domain. If U_{YY} changes sign it means that there are two regions with opposite vorticity gradient. When $U_{YY}(U-U_I)<0$ it means that one of the regions is been convected with a velocity smaller than its local flow velocity (against its direction of propagation).







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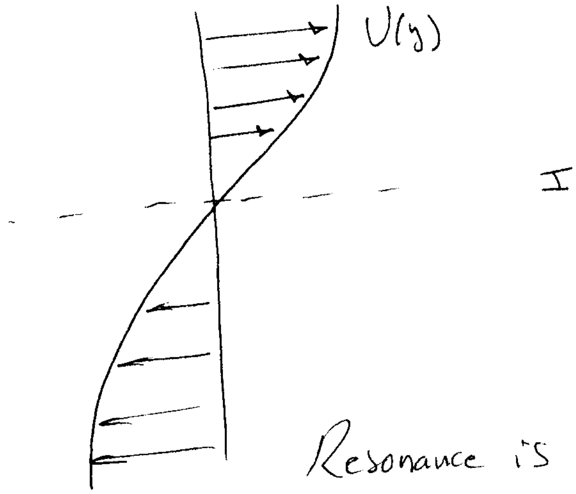

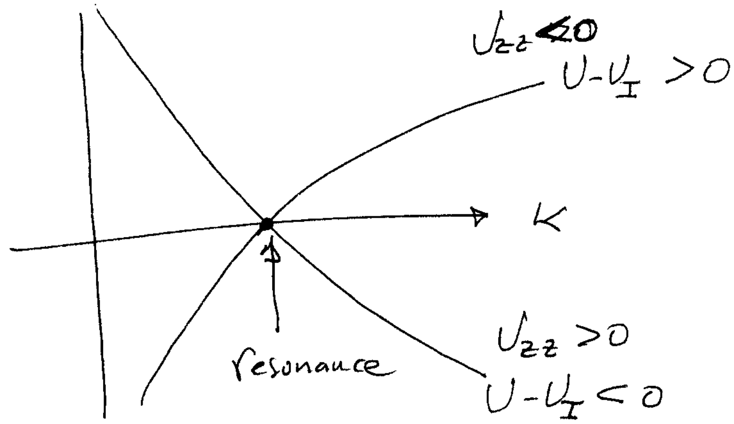

Resonance is possible if the phase speed of shear waves are equal. For that at least one of the shear waves have to be convected by the local velocity against its direction of propagation