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Example: Simple shear with no density difference

$$U(y) = \begin{cases} \Omega_0 y & y>0 \\ 0 & y<0 \end{cases}$$

Taylor - Goldstein equation can be written as:
(Rayleigh)

$$v_{yy} - \left[\frac{U_{yy}}{U-c} + k^2 - \frac{N^2}{(U-c)^2}\right] v = 0 \qquad N = \sqrt{\frac{-g}{\rho_0}\frac{d\rho_0}{dz}} > 0$$

Brunt - Vaisala frequency

$U_{yy} = 0$

$N = 0$

$$v_{yy} - k^2 v = 0$$

$$v = \begin{cases} A e^{-ky} & y>0 \\ B e^{ky} & y<0 \end{cases}$$

v is the velocity normal to the free stream (interface between shear and no shear): $v = \frac{D\eta}{Dt} = \eta_t + U \cdot \eta_x$

η is the vertical displacement of the interface.

since v is continuous: $A = B$

The boundary condotion (normal) at the interface is

- Total pressure is constant across the interface

$$P(0^-) - g\rho_1 \eta = P(0^+) - g\rho_2 \eta$$

linearized at $y=0 \Rightarrow P = \frac{\rho\, ik(U-c)u + U_y v}{-ik}$ and, from continuity, $iku + v_y = 0 \Rightarrow P = i\rho\left[-\frac{(U-c)v_y}{k} + \frac{U_y v}{k}\right]$

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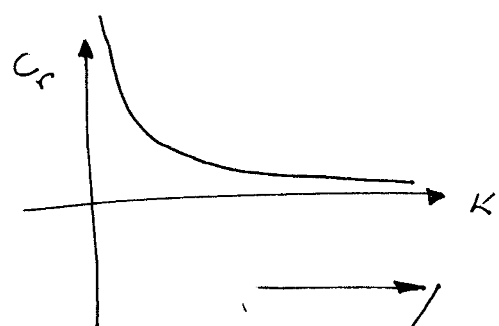
So $\cancel{i\rho}\left[-\frac{(\Omega_0 y-c)V_y}{K}+\frac{\Omega_0 V}{K}\right]-\cancel{\rho g\eta}=\cancel{i\rho}\left[-\frac{(0-c)V_y}{K}+0\right]-\cancel{\rho g\eta}$

Since $V=\begin{cases} A e^{-ky} & y>0 \\ B e^{ky} & y<0 \end{cases}$

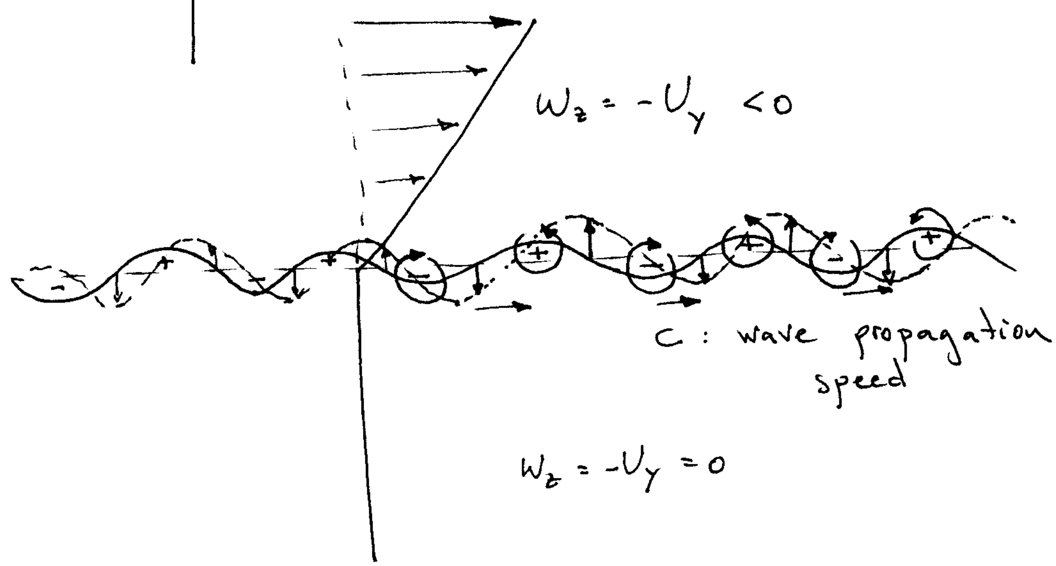
$$-\frac{(\Omega_0 y-c)\cancel{B}(-k)e^{-ky}}{\cancel{K}}+\frac{\Omega_0 B e^{ky}}{\cancel{K}}=+\frac{c\,\cancel{B}(+k)e^{+ky}}{K}$$

particularizing at $y=0$:

$$-cK+\Omega_0=cK \Rightarrow \boxed{c=\frac{\Omega_0}{2K}}$$ Dispersion relation



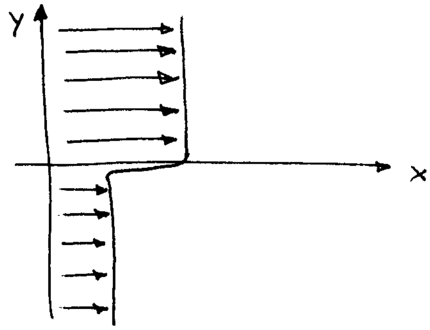
$c_i=0 \Rightarrow$ neutrally stable periodic waves



A displacement of the interface has a perturbation of the vorticity field associated with it.

Kelvin-Helmholtz instability

$$U(y) = \begin{cases} U_1 & y>0 \\ U_2 & y<0 \end{cases}$$



The Rayleigh (or Taylor-Goldstein) equation reduces to

$$v_{yy} - k^2 v = 0 \qquad \text{because} \quad U_{yy} = 0 \quad \text{and} \quad N = 0$$

$$v = \begin{cases} A e^{-ky} & y>0 \\ B e^{ky} & y<0 \end{cases}$$

The dynamic boundary condition at the interface (equilibrium of normal stresses) results in

$$P(0^+) - \rho_1 g \eta = P(0^-) - \rho_2 g \eta$$

Where η is the vertical displacement of the interface

$$v\Big|_{y=0} = \frac{D\eta}{Dt} = \eta_t + U\eta_x \; ; \quad \eta = \eta_0 e^{ik(x-ct)}$$

$$-ikc\,\eta + ik\underset{\substack{U_1 \; y\to 0^+ \\ U_2 \; y\to 0^-}}{U}\eta = \begin{cases} A & y\to 0^+ \\ B & y\to 0^- \end{cases}$$

$$\eta = \frac{A}{ik(U_1-c)} = \frac{B}{ik(U_2-c)}$$

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$$\underbrace{i\left[-\frac{(U_1-c)}{k}V_y(0^+)+\frac{\cancel{U_Y}^{\,0}V(0^+)}{k}\right]}_{P(0^+)}-\rho_1 g\frac{A}{ik(U_1-c)}=\overbrace{i\rho_2\left[-\frac{U_2-c}{k}V_y(0^-)+\frac{\cancel{U_Y}^{\,0}V(0^-)}{k}\right]}^{P(0^-)}-\rho_2 g\frac{B}{ik(U_2-c)}$$

$$i\rho_1\,\frac{+(U_1-c)}{\cancel{k}}(+k)\cancel{A}e^{-kY}\Big|_{Y\to0^+}-\rho_1 g\frac{\cancel{A}}{\cancel{i}k(U_1-c)}=i\rho_2\left[-\frac{(U_2-c)}{\cancel{k}}\cancel{k}\cancel{B}e^{kY}\right]_{Y\to0^-}-\rho_2 g\frac{\cancel{B}}{\cancel{i}k(U_2-c)}$$

$$-\cancel{i}k(U_1-c)\cancel{i}\rho_1(U_1-c)-\rho_1 g=\cancel{i}k(U_2-c)\cancel{i}\rho_2\left[+(U_2-c)\right]-\rho_2 g$$

$$k\left[\rho_1(U_2-c)^2+\rho_2(U_1-c)^2\right]+g(\rho-\rho)=0$$

$$c^2(\rho_1+\rho_2)-2c(\rho_1U_1+U_2\rho_2)+\frac{g}{k}(\rho-\rho)+\rho_1U_1^2+\rho_2U_2^2=0$$

$$c=\frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2}\pm\sqrt{\frac{\rho_1^2U_1^2+\rho_2^2U_2^2+2\rho_1\rho_2U_1U_2-\frac{g}{k}(\rho_1-\rho_2)(\rho_1+\rho_2)}{(\rho_1+\rho_2)^2}-\frac{(\rho_1U_1^2+\rho_2U_2^2)(\rho_1+\rho_2)}{(\rho_1+\rho_2)^2}}$$

$$c=\frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2}\pm\sqrt{\frac{\cancel{\rho_1^2U_1^2}+\cancel{\rho_2^2U_2^2}+2\rho_1\rho_2U_1U_2-\cancel{\rho_1^2U_1^2}-\cancel{\rho_2^2U_2^2}-\rho_1\rho_2U_1^2-\rho_1\rho_2U_2^2}{(\rho_1+\rho_2)^2}-\frac{g}{k}\frac{\rho_1-\rho_2}{\rho_1+\rho_2}}$$

$$\boxed{c=\frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2}\pm\sqrt{\frac{g}{k}\frac{\rho_2-\rho_1}{\rho_2+\rho_1}-\frac{\rho_1\rho_2(U_1-U_2)^2}{(\rho_2+\rho_1)^2}}}$$

Unstable if: $\frac{g}{k}\frac{\rho_2-\rho_1}{\rho_2+\rho_1}<\frac{\rho_1\rho_2}{(\rho_2+\rho_1)^2}(U_1-U_2)^2$

if $\rho_1 = \rho_2 \Rightarrow$ always unstable, independent of k
viscosity or surface tension reintroduce the dependency on k.

if $U_1 = U_2 = 0 \Rightarrow$ Rayleigh-Taylor: unstable if $\rho_2 < \rho_1$ (with surface tension) for any k

if $\rho_1 \neq \rho_2$ and $U_1 \neq U_2$: unstable to high values of k (short wavelengths)
This is unphysical: surface tension or viscosity damp out the infinitesimally small waves ($k \uparrow\uparrow$)