Taylor - Couette Instability

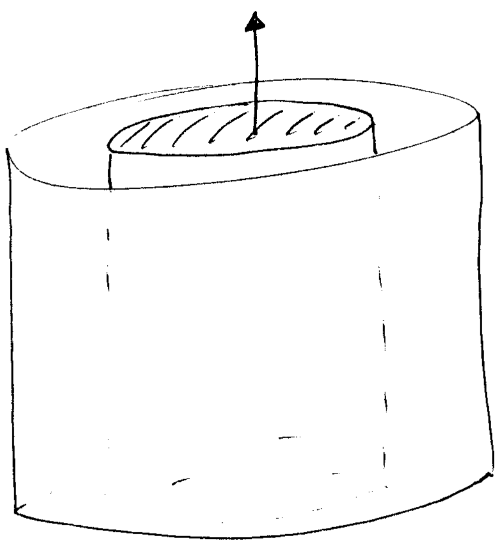
Governing equations:

$$\frac{\partial V_r}{\partial t} + (\vec{V}\cdot\nabla)V_r - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\nabla^2 V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r^2}\right)$$

$$\frac{\partial V_\theta}{\partial t} + (\vec{V}\cdot\nabla)V_\theta + \frac{V_r V_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta} + \nu\left(\nabla^2 V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2}\right)$$

$$\frac{\partial V_z}{\partial t} + (\vec{V}\cdot\nabla)V_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 V_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$



Inner cylinder has $r = R_1$ and $\vec{\omega} = \Omega_1 \vec{k}$

Outer cylinder has $r = R_2$ and $\vec{\omega} = \Omega_2 \vec{k}$

Equilibrium state is:

- Steady $\frac{\partial}{\partial t}$
- $V_r = V_z = 0$
- $V_\theta = A\cdot r + \frac{B}{r}$

Where $A = \dfrac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$

$$B = \frac{(\Omega_1 - \Omega_2)\, R_1^2 R_2^2}{R_2^2 - R_1^2}$$

$r = R_2 r^*$

$t = \dfrac{t^*}{\Omega_2}$

$\vec{V} = R_2 \Omega_1 \vec{V}^*$

$P = \rho_0 (R_2 \Omega_2)^2 P^*$

Equations become

$$R_2 \Omega_2^2 \frac{\partial V_r^*}{\partial t^*} + R_2 \Omega_2^2 (\vec{V}^* \cdot \nabla^*) V_r^* - R_2 \Omega_2^2 \frac{V_\theta^{*2}}{r^*} =$$

$$= - R_2 \Omega_2^2 \frac{\partial P^*}{\partial r^*} + \nu \frac{\Omega_2}{R_2} \left(\nabla^{*2} V_r^* - \frac{2}{r^{*2}} \frac{\partial V_\theta^*}{\partial \theta} - \frac{V_r^*}{r^{*2}} \right)$$

$$\frac{\partial V_r^*}{\partial t^*} + (\vec{V}^* \cdot \nabla^*) V_r^* - \frac{V_\theta^{*2}}{r^*} = - \frac{\partial P^*}{\partial r^*} + \frac{\nu}{\Omega_2 R_2^2} \left(\nabla^{*2} V_r^* - \frac{2}{r^{*2}} \frac{\partial V_\theta^*}{\partial \theta} - \frac{V_r^*}{r^{*2}} \right)$$

$$\boxed{\frac{\Omega_2 R_2^2}{\nu} = T_a \quad \text{(Taylor number)}}$$

only non dimensional parameter

Linearize the equations:

$\vec{V}^* = (u, V_\theta + v, w)$

$P^* = (P + p)$

$$\frac{\partial u}{\partial t} + V_\theta \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{V_\theta^2 + 2 V_\theta v + \cancel{v^2}}{r} = - \frac{\partial P}{\partial r} - \frac{\partial p}{\partial r} + \frac{1}{T_a} \left(\nabla^2 u - \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right)$$

linearize

$$\frac{\partial v}{\partial t} + V_\theta \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u V_\theta}{r} + \cancel{\frac{uv}{r}} + u \frac{\partial V_\theta}{\partial r} = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{T_a} \left(\nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2} \right) + \cancel{\frac{1}{T_a} \left(\nabla^2 V_\theta - \frac{V_\theta}{r^2} \right)}$$

linearize

cancel from solution of base state

cancel from solution of base state.

$$\frac{\partial w}{\partial t} + V_\theta \frac{1}{r} \frac{\partial w}{\partial \theta} = - \frac{\partial p}{\partial z} + \frac{1}{T_a} (\nabla^2 w)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r u) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

Decompose in normal modes:

$$u = u(r)\, e^{i(n\theta + mz) + \sigma t} \qquad P = P(r)\, e^{i(n\theta + mz) + \sigma t}$$

$$v = v(r)\, e^{i(n\theta + mz) + \sigma t}$$

$$w = W(r)\, e^{i(n\theta + mz) + \sigma t}$$

n: represents azimuthal waves

$n = 0$ is for axisymmetric perturbations which grow fastest (Squire's Theorem)

m: represents axial waves. m **has** to be real so that the perturbations are bounded in space, as $z \to \pm\infty$

$$\sigma v + u \underbrace{\frac{\partial V_\theta}{\partial r}}_{u\left(A - \frac{B}{r^2}\right)} + \underbrace{V_\theta \frac{1}{r}}_{A + \frac{B}{r^2}} i n v + \underbrace{u \frac{V_\theta}{r}}_{u\left(A + \frac{B}{r^2}\right)} = -\frac{1}{r} i n p + \frac{1}{Ta}\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) v + (in)^2 v + (im)^2 v + \frac{2}{r^2} i n u\right]$$

$$\sigma u + \left(A + \frac{B}{r^2}\right) i n u - 2\left(A + \frac{B}{r^2}\right) v = -\frac{\partial P}{\partial r} + \frac{1}{Ta}\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) u + (in)^2 u + (im)^2 u - \frac{2}{r^2} i n v\right]$$

$$\sigma W + \left(A + \frac{B}{r^2}\right) i n W = -imp + \frac{1}{Ta}\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) W + (in)^2 W + (im)^2 W\right]$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + i n v + i m W = 0$$

Make $n = 0$, then get rid of W from the continuity equation

$$W = -\frac{1}{im}\frac{1}{r}\frac{\partial (ru)}{\partial r}$$

(21)

$$im\sigma u - 2\left(A+\frac{B}{r^2}\right) imv = -im\cancel{\frac{\partial p}{\partial r}} + im\frac{1}{Ta}\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-m^2\right)u$$

$$- \quad \sigma\frac{\partial w}{\partial r} = -im\cancel{\frac{\partial p}{\partial r}} + \frac{1}{Ta}\frac{d}{dr}\left[\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)w - m^2 w\right]$$

$$-2\left(A+\frac{B}{r^2}\right) imv = \left[\frac{1}{Ta}\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-m^2\right) - \right] u + \frac{1}{Ta}\left[\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)\frac{dw}{dr} - m^2\frac{dw}{dr}\right] - \sigma\frac{dw}{dr}$$

substituting $\frac{dw}{dr} = -\frac{1}{im}\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = -\frac{1}{im}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)u$

and multiplying by im to make it an equation with real terms (no imaginary)

$$2\left(A+\frac{B}{r^2}\right)m^2 v = \left[\frac{1}{Ta}\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-m^2\right)-\sigma\right](-m^2 u) + \left[\frac{1}{Ta}\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-m^2\right)-\sigma\right]\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)u$$

From the θ-momentum equation we get that

$$\sigma v + 2uA = \frac{1}{Ta}\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-m^2\right)v$$

so $$u = \frac{1}{2ATa}\left(\frac{d}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-m^2-\sigma\right)v$$

and substituting in the previous expressions we get:

$$\frac{1}{Ta}\left(D^2-m^2-\sigma\right)\left(D^2-m^2\right)\frac{1}{2ATa}\left(D^2-m^2-\sigma\right)v = 2\left(A+\frac{B}{r^2}\right)m^2 v$$

To get the stability we can find the border between stability $\sigma_r < 0$ and instability $\sigma_r > 0$.
To do that we try $\sigma = 0 \Rightarrow$

$$\left[(D^2 - m^2)(D^2 - m^2)(D^2 - m^2) - 4\left(A + \frac{B}{r^2}\right) A \, m^2 Ta^2 \right] v = 0$$

with b.c. $v(r=1) = 0$ and $v\left(r = \frac{R_1}{R_2}\right) = 0$

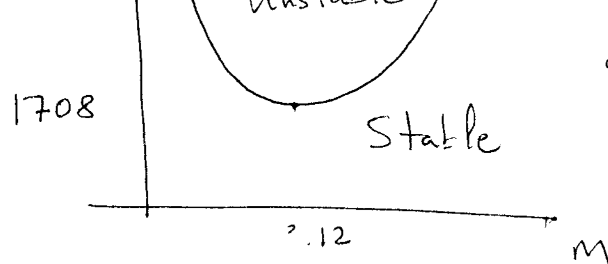
outer wall   inner wall

Since we have a 6^{th} order ODE we need 6 b.c.s

$$u = 0 \Rightarrow \left(\frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - m^2\right) v = 0 \quad \text{at } r \; \frac{R_1}{R_2}, 1$$

$$W = 0 \Rightarrow \left(\frac{d}{dr} + \frac{1}{r}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - m^2\right) v = 0 \quad \text{at } r = \frac{R_1}{R_2}, 1$$

We have a 6^{th} order homogeneous O.D.E with 6 homogeneous b.c.s $\Rightarrow$ Eigenvalue problem $Ta(m)$
We can find the values of the Taylor number for which different wavelengths (m) are neutrally stable.



The flow will become unstable as soon as there is one wavelength that has positive growth rate.