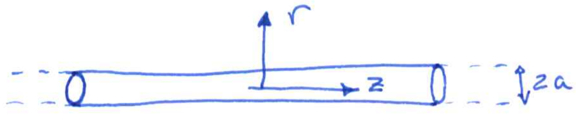
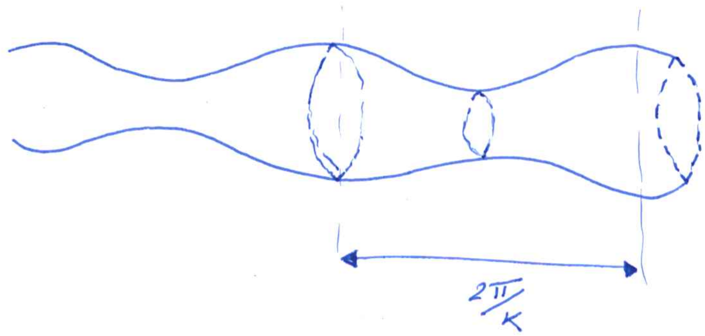
Rayleigh-Plateau Instability

Capillary instability of a liquid cylinder



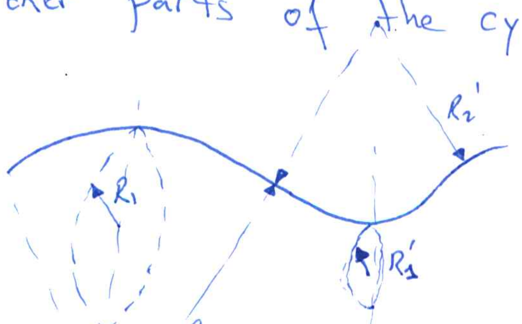
Fluid cylinder of radius a with surface tension σ

Equilibrium state: $\vec{v}=0$; $P = P_\infty + \frac{\sigma}{a}$ inside the liquid



If the wavelength is long compared to the cylinder's radius $(k \cdot a \ll 1)$ the azimuthal curvature dominates over the longitudinal curvature.

The reduction in the radius drives an increase in pressure which is much more significant than that induced by the longitudinal curvature. A pressure gradient is established that drives the fluid from the narrow region towards the thicker parts of the cylinder, amplifying the instability.



$$P_\infty + \sigma\left(\frac{1}{R_1'} - \frac{1}{R_2'}\right)$$

$$\frac{1}{R_2'} + \frac{1}{R_2} \simeq k \ll \frac{1}{a}$$

$$P_\infty + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\frac{\Delta P}{\lambda/2} = \sigma\left(\frac{1}{R_1'} - \frac{1}{R_1}\right) - \left(\frac{1}{R_2'} + \frac{1}{R_2}\right) > 0$$

Basic equations: Incompressible, inviscid Navier-Stokes.

$$\rho \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla p$$

$$\nabla\cdot\vec{v} = 0$$

Since the basic state is $\vec{V}=0$ and $\nabla P = 0$ these are already the linearized equations.

The boundary conditions are: • $v_r = \frac{D\eta}{Dt}$ where $\eta(\theta,z,t)$ is the position of the interface.

• $P = P_\infty + \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$

The radius of curvature $\frac{1}{R_1}, \frac{1}{R_2}$ can be related to the second derivative of the interface position by:

$r = \eta(\theta,z,t) \Rightarrow F(r,\theta,z,t) = 0$

normal to that surface is $\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{\left(1, -\frac{1}{r}\frac{\partial\eta}{\partial\theta}, -\frac{\partial\eta}{\partial z}\right)}{\sqrt{1+\left(\frac{1}{r}\frac{\partial\eta}{\partial\theta}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2}}$

The curvature of the surface $\frac{1}{R_1}+\frac{1}{R_2}$ can be written as $\nabla\cdot\vec{n}$. But before we evaluate that expression we can linearized it for small deformations: $\eta = a+\eta'$ where $\eta' << a$: $\vec{n} = \left(1, -\frac{1}{r}\frac{\partial\eta'}{\partial\theta}, -\frac{\partial\eta'}{\partial z}\right)$

$$\nabla\cdot\vec{n} = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot 1) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial\eta'}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial\eta'}{\partial z}\right)$$

$\nabla\cdot\vec{n} = \frac{1}{r} + \left(\frac{-1}{r^2}\right)\frac{\partial^2\eta'}{\partial\theta^2} - \frac{\partial^2\eta'}{\partial z^2}$ and evaluate it at $r=\eta=a+\eta'$

$$\nabla \cdot \vec{n} = \frac{1}{a+\eta'} - \frac{1}{(a+\eta')^2} \frac{\partial^2 \eta'}{\partial \theta^2} - \frac{\partial^2 \eta'}{\partial z^2}$$

linearizing again: $\nabla \cdot \vec{n} = \frac{1}{a}\left(1 - \frac{\eta'}{a}\right) - \frac{1}{a^2} \frac{\partial^2 \eta'}{\partial \theta^2} - \frac{\partial^2 \eta'}{\partial z^2}$

$$P = \bar{P} + p' = P_\infty + \sigma \nabla \cdot \vec{n} = P_\infty + \sigma \frac{1}{a} + p'$$

$$p' = \sigma \left[\cancel{\frac{1}{a}}\left(\cancel{1} - \frac{\eta'}{a}\right) - \frac{1}{a^2} \frac{\partial^2 \eta'}{\partial \theta^2} - \frac{\partial^2 \eta'}{\partial z^2} \right] - \cancel{\sigma \frac{1}{a}}$$

$$p' = -\sigma \left(\frac{\eta'}{a^2} + \frac{1}{a^2} \eta'_{\theta\theta} + \eta'_{zz} \right) \quad \text{at } r = a$$

$$v_r = \frac{\partial \eta}{\partial t} + \cancel{V_r \frac{\partial \eta}{\partial r} + V_\theta \frac{1}{r} \frac{\partial \eta}{\partial \theta} + V_z \frac{\partial \eta}{\partial z}} \rightarrow 0$$

no velocity in the basic state

Linearized Navier-Stokes

$$\rho \frac{\partial \vec{v}}{\partial t} + \cancel{(\vec{v} \cdot \nabla)\vec{v}}^{\text{linearized}} = -\nabla p' \rightarrow \rho \nabla \cdot \frac{\partial \vec{v}}{\partial t} = -\nabla \cdot \nabla p'$$

$$\nabla \cdot \vec{v} = 0 \quad \rightarrow \frac{\partial}{\partial t}(\nabla \cdot \vec{v}) = 0$$

$$\boxed{\nabla^2 p' = 0}$$

Laplace's eq.

p' acts as a velocity potential

Normal modes: $(\vec{v}, p', \eta') = (\vec{v}(r), p(r), \eta)\, e^{i(kz + n\theta) + \sigma t}$

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r}\right) + \frac{1}{r^2} \frac{\partial p'}{\partial \theta^2} + \frac{\partial^2 p'}{\partial z^2} = 0$$

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{dp}{dr}\right) + \frac{1}{r^2} (in)^2 p + (ik)^2 p = 0$$

$$\frac{d^2 p}{dr^2} + \frac{1}{r} \frac{dp}{dr} - \left(k^2 + \frac{n^2}{r^2}\right) p = 0$$

Modified Bessel equation

$$p = A\, I_n(kr) + \cancel{B K_n(kr)} \rightarrow 0$$

K_n not regular at the origin $r = 0$

The boundary conditions are:

$$P = -\sigma\left[\frac{\eta}{a^2} + \frac{(in)^2}{a^2}\eta + (ik)^2\eta\right] \quad \text{at } r=a$$

$$v_r = \gamma\eta$$

and we know that $\rho\gamma v_r = -Ak I_n'(kr)$

derivative of the modified Bessel function of the first kind I_n

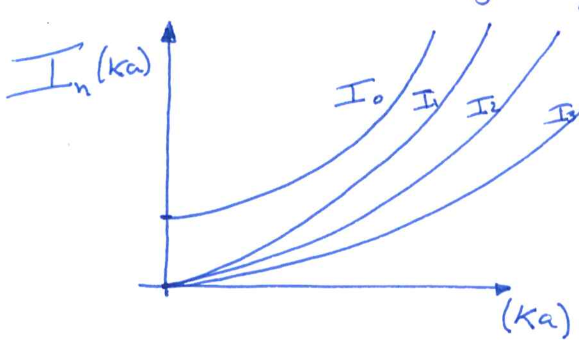
so at $r=a$:

$$\rho\gamma^2\eta = -Ak I_n'(ka)$$

and also $-\sigma\left(\frac{1}{a^2} - \frac{n^2}{a^2} - k^2\right)\eta = A I_n(ka)$

$$+\sigma\left(\frac{1}{a^2} - \frac{n^2}{a^2} - k^2\right)\frac{+Ak I_n'(ka)}{\rho\gamma^2} = A I_n(k\cdot a)$$

$$\gamma^2 = \frac{\sigma}{\rho a^3}\left[\frac{(ka)I_n'(ka)}{I_n(k\cdot a)}\right]\left[1 - n^2 - (ka)^2\right]$$



$I_n(ka) > 0$

$I_n'(ka) > 0$

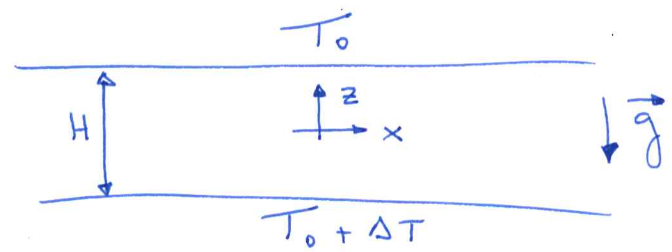
- if n is equal no 1 or greater $\gamma^2 < 0 \Rightarrow \gamma = i\omega$ and the cylinder is neutrally stable to all non-axisymmetric perturbations.

- if $n=0$ stability depends on the relative value of $ka \lessgtr 1$. If $ka<1$ (long wavelength, small k) the cylinder is unstable $\Rightarrow$ growth rate $\gamma>0$
If $ka<1$ (short wavelength, high k) the cylinder is stab[le] even to axisymmetric pertarbations $\gamma<0$

For $n=0$, the fastest growing mode is $ka=0.6570$ and the growth rate is $\gamma\left(\frac{a^3\rho}{\sigma}\right)^{1/3} = 0.3433$.

This information, due to Lord Rayleigh, can be used to predict the size of the resulting droplets from the instability breaking the cylinder into small spherical drops.

RAYLEIGH - BENARD CONVECTION



We use incompressible assumption with Boussinesq approximation for the influence of gravity on density difference.

$$\underbrace{\rho = \rho_0 \left[1 - \alpha(T - T_0)\right]}_{\substack{\text{linear expression} \\ \Delta T \ll T_0}}$$

where $\alpha = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial T}$ is the fluid's coefficient of thermal expansion

- $$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{v}$$

Boussinesq makes $\rho = \rho_0$ everywhere except in the gravity term: $\rho_0 \vec{g}\left[1 - \alpha(T - T_0)\right]$

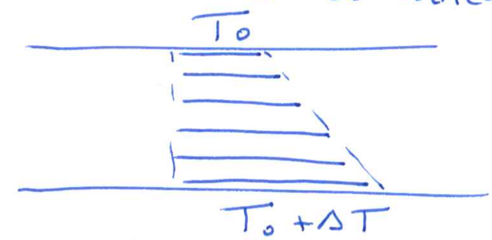
- $\nabla \cdot \vec{v} = 0$

- $$\frac{D\rho}{Dt} = \kappa \nabla^2 \rho \quad \text{or equivalently} \quad \frac{DT}{Dt} = \underset{\substack{\uparrow \\ \text{coefficient of thermal diffusion}}}{\kappa_T} \nabla^2 T$$

Equilibrium state

$$\vec{v} = \vec{0}$$
$$\rho = \rho_0 (1 - \beta z); \quad T = T_0 - \frac{\Delta T}{H}\left(z - \frac{H}{2}\right)$$
$$\nabla p = \rho \vec{g}$$

In the equilibrium state, there is a constant heat flux $\vec{q} = k_T \frac{\Delta T}{H}\vec{k}$ which establishes a linear temperature distribution



Non-dimensionalization:

$$z = H z^* \qquad P = \left(\rho_0 \kappa^2/H^2\right) p^*$$

$$t = \frac{H^2}{\kappa} t^* \qquad T = \Delta T\, T^* + T_0$$

$$T^* = \frac{T - T_0}{\Delta T}$$

$$\vec{v} = - \vec{v}^*$$

$$\nabla \cdot \vec{v}^* = 0$$

$$\frac{\cancel{\Delta T}}{H^2/\kappa}\frac{\partial T^*}{\partial t^*} + \frac{\frac{\kappa}{H}\cancel{\Delta T}}{H}\left(\vec{v}^* \cdot \nabla^*\right) T^* = \kappa \frac{\cancel{\Delta T}}{H^2}\nabla^{*2} T^*$$

$$\frac{\kappa/H}{H^2/\kappa}\frac{\partial \vec{v}^*}{\partial t^*} + \frac{(\kappa/H)^2}{H}\left(\vec{v}^* \cdot \nabla^*\right)\vec{v}^* = -\frac{1}{\rho_0}\frac{\rho_0 \kappa^2/H^2}{H}\nabla^* p^* + \frac{\rho}{\rho_0}\vec{g} + \nu\frac{\kappa/H}{H^2}\nabla^{*2}\vec{v}^*$$

$$\frac{\partial \vec{v}^*}{\partial t^*} + \left(\vec{v}^* \cdot \nabla^*\right)\vec{v}^* = -\nabla^* p^* + \frac{H^3}{\kappa^2}\frac{\rho_0\left(1 - \alpha\Delta T\, T^*\right)}{\rho_0} g\vec{k} + \frac{\nu}{\kappa}\nabla^{*2}\vec{v}^*$$

Basic state: $$0 = -\nabla^* \bar{P}^* + \frac{H^3}{\kappa^2}\frac{\rho_0\left[1 + \alpha\Delta T\left(\frac{z - H/2}{H}\right)\right]}{\rho_0} g\vec{k}$$

Perturbation pressure (p): $P^* = \bar{P}^* + p'$ and temperature $T^* = -\frac{z - H/2}{H} + T'$

$$\frac{\partial \vec{v}^*}{\partial t^*} + \left(\vec{v}^* \cdot \nabla^*\right)\vec{v}^* = -\nabla^* p' + \left(\frac{H^3}{\kappa^2}\alpha\Delta T g\right)T'\vec{k} + \frac{\nu}{\kappa}\nabla^{*2}\vec{v}^*$$

$$\underbrace{\frac{g\alpha\Delta T H^3}{\kappa\nu}}_{\text{Rayleigh number}} \cdot \underbrace{\frac{\nu}{\kappa}}_{\text{Prandtl number}}$$

$\frac{\nu}{\kappa} \rightarrow$ Prandtl number

Dropping stars

- $$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla p' + P_r R_a T' \vec{k} + P_r \nabla^2 \vec{v}$$
- $$\nabla\cdot\vec{v} = 0$$
- $$\frac{\partial T'}{\partial t} + (\vec{v}\cdot\nabla)\left(T' - \frac{z - H/2}{H}\right) = \nabla^2\left(T' - \frac{z-H/2}{H}\right)$$

Linearizing:

$$\left.\begin{aligned}
&\bullet\ \frac{\partial \vec{v}}{\partial t} = -\nabla p' + P_r R_a T' \vec{k} + P_r \nabla^2 \vec{v} \\
&\bullet\ \nabla\cdot\vec{v} = 0 \\
&\bullet\ \frac{\partial T'}{\partial t} - v_z \frac{\partial}{\partial (z/H)}\left(-\frac{z}{H} + \frac{1}{2}\right) = \nabla^2 T'
\end{aligned}\right\}$$

5 eq with 5 unknowns
$\vec{v}, P', T'$

$$\frac{\partial v_x}{\partial t} = -\frac{\partial P'}{\partial x} + P_r \nabla^2 v_x \quad \frac{\partial}{\partial y}$$

$$\frac{\partial v_y}{\partial t} = -\frac{\partial P'}{\partial y} + P_r \nabla^2 v_y \quad \frac{\partial}{\partial x}$$

$\ominus \rightarrow$
$$\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) = P_r \nabla^2\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right)$$

$\frac{\partial}{\partial x}$ $\oplus$ $\frac{\partial}{\partial y}$
$$\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = -\nabla_H^2 P + P_r \nabla^2\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

continuity

$$\frac{\partial}{\partial t}\left(-\frac{\partial v_z}{\partial z}\right) = -\nabla_H^2 P + P_r \nabla^2\left(-\frac{\partial v_z}{\partial z}\right)$$

Take $\frac{\partial}{\partial z}$ of this equation to get:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 v_z}{\partial z^2}\right) = \nabla_H^2\left(\frac{\partial P'}{\partial z}\right) + P_r \nabla^2\left(\frac{\partial^2 v_z}{\partial z^2}\right)$$

Taking $\nabla_H^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ of the z-momentum equation

$$\frac{\partial}{\partial t}\left(\nabla_H^2 v_z\right) = -\nabla_H^2\left(\frac{\partial P'}{\partial z}\right) + R_a P_r \nabla_H^2 T' + P_r \nabla_H^2 \nabla^2 v_z$$

Eliminating the pressure by adding the 2 equations together.

$$\frac{\partial}{\partial t}\underbrace{\left(\nabla_H^2 V_z + \frac{\partial^2 V_z}{\partial z^2}\right)}_{\nabla^2 V_z} = Ra\,Pr\,\nabla_H^2 T' + Pr\,\nabla^2\underbrace{\left(\nabla_H^2 V_z + \frac{\partial^2 V_z}{\partial z^2}\right)}_{\nabla^2 V_z}$$

We have: $Ra\,Pr\,\nabla_H^2 T' = -Pr\,\nabla^4 V_z + \frac{\partial}{\partial t}(\nabla^2 V_z)$

and from the energy equation: $\frac{\partial T'}{\partial t} - \nabla^2 T' = V_z$

2 eq. and 2 unknowns.

If we take $\left(\frac{\partial}{\partial t} - \nabla^2\right)$ of the first one and substitute in the second one:

$$Ra\,Pr\,\nabla_H^2 \underbrace{\left(\frac{\partial}{\partial t} - \nabla^2\right)T'}_{V_z} = \left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - Pr\,\nabla^2\right)\nabla^2 V_z$$

$$\boxed{Ra\,Pr\,\nabla_H^2 V_z = \left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - Pr\,\nabla^2\right)\nabla^2 V_z}$$

If we assume solutions by normal modes:

$$V_z = V_z(z)\, e^{i(kx+my)+\sigma t}$$

$$= \left\{\sigma - \left[(ik)^2 + (im)^2\right]\right. \quad \left\{\sigma - Pr\left[(ik)^2 + (im)^2 + \frac{\partial^2}{\partial z^2}\right]\right\}\left[(ik)^2 + (im)^2 + \frac{\partial^2}{\partial z^2}\right] V_z$$

$$\left[\left(\sigma + k^2 - \frac{d^2}{dz^2}\right)\left(\frac{1}{Pr}\sigma + k^2 - \frac{d^2}{dz^2}\right)\left(\frac{d^2}{dz^2} - k^2\right) + Ra\,k^2\right] V_z = 0$$

6^{th} order homogeneous equation

- At $z = 0, 1$ $T' = 0$ fixed temperature at the boundaries
  - if the boundaries are rigid (solid boundaries): $V_x = V_y = V_z = 0$ at $z =$

    and, because the boundaries extend along the x and y directions

$$\frac{\partial V_x}{\partial x} = \frac{\partial V_y}{\partial y} = 0 \Rightarrow \underline{\frac{\partial V_z}{\partial z} = -\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} = 0 \text{ at } z =}$$

if the boundaries are free surfaces, then the condition is free stress: $\frac{\partial V_x}{\partial z} = \frac{\partial V_y}{\partial z} = 0$ at $z=$

from continuity: $\frac{\partial^2 V_z}{\partial z^2} = -\frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial z}\right) = 0$

and $W = 0$

For this last case, the form of $V_z(z)$ that satisfies the boundary conditions is $V_z(z) = \sin(n\pi z)$

$$(\sigma + k_n^2)\left(\frac{1}{P_r}\sigma + k_n^2\right)k_n^2 - R_a k^2 = 0$$

where $k_n^2 = k^2 + (n\pi)^2$

$$\sigma^2 + (1+P_r)\sigma k_n^2 + P_r k_n^4 - R_a P_r \frac{k^2}{k_n^2} = 0$$

$$\sigma = -\frac{1}{2}(1+P_r)k_n^2 \pm \sqrt{\frac{1}{4}(1+P_r)^2 k_n^4 - P_r k_n^4 + P_r R_a \frac{k^2}{k_n^2}}$$

$$\sigma = -\frac{1}{2}(1+P_r)k_n^2 \pm \sqrt{\underbrace{\frac{1}{4}(1-P_r)^2 k_n^2}_{>} + \underbrace{P_r R_a \frac{k^2}{k_n^2}}_{>0}}$$

if $R_a > 0$
$\Delta T > 0$

σ is always real

instability can be described by the boundary $\sigma = 0$ (exchange of instabilities)

if $\underline{\sigma = 0} \Rightarrow R_a \cancel{P_r} \frac{k^2}{k_n^2} = \cancel{P_r} k_n^4$

$$\boxed{R_a = \frac{k_n^6}{k^2}}$$

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minimum Ra for the first mode $n=1$:

$$Ra = \frac{(K^2 + n^2\pi^2)^3}{K^2}$$

$$\left.\frac{\partial Ra}{\partial K^2}\right|_{n=1} = \frac{3(K^2+\pi^2)^{\cancel{2}} K^2 - (K^2+\pi^2)^{\cancel{3}}}{K^4} = 0$$

$$3K^2 - K^2 + \pi^2 = 0 \qquad \boxed{K^2 = \frac{\pi^2}{2}}$$

$$Ra_c = \frac{(\pi^2/2 + \pi^2)^3}{\pi^2/2} = \frac{\frac{27}{8}\pi^6}{\pi^2/2} = \frac{27}{4}\pi^4 \simeq 657.2$$

if $Ra > Ra_c \Rightarrow$ the most unstable perturbation (K) consistent with a mode 1 instability will have positive growth rate.

For the rigid boundaries case:

We can write the general solution as $V_z = e^{qz}$

$$\left(\sigma + K^2 - q^2\right)\left(\frac{1}{P_r}\sigma + K^2 - q^2\right)\left(q^2 - K^2\right) + Ra\, K^2 = 0$$

Applying the principle of exchange of stabilities $(\sigma = 0)$

$$(K^2 - q^2)(K^2 - q^2)(q^2 - K^2) + Ra\, K^2 = 0$$

$$(q^2 - K^2)^3 = -Ra\, K^2$$

$$q^2 = -K^2\left[\left(\frac{Ra}{K^4}\right)^{1/3} - 1\right] \rightarrow \begin{cases} \pm iK\sqrt{\left(\frac{Ra}{K^4}\right)^{1/3} - 1} \\ \pm K\sqrt{1 + \frac{1}{2}\left(\frac{Ra}{K^4}\right)^{1/3}(1 + i\sqrt{3})} \\ \pm K\sqrt{1 + \frac{1}{2}\left(\frac{Ra}{K^4}\right)^{1/3}(1 - i\sqrt{3})} \end{cases}$$