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VORTICITY DYNAMICS

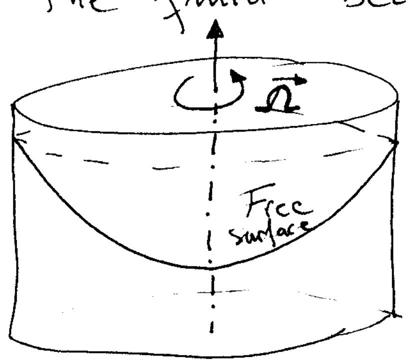
What is vorticity? $\vec{\omega} = \nabla \times \vec{v}$

From the break-up of the motion of a d element into:

- Translation
- Solid body rotation
- Dilatation (extensional strain)
- Tangential deformation (shear strain)

We found that vorticity $\vec{\omega}$ is twice the rate of rotation of the fluid element: $\vec{\omega} = 2\vec{\Omega}$

Rotation (and therefore vorticity) do not contribute to the shear stresses that develop in the fluid because of its rate of deformation.



Continuity

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \cancel{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

Assuming plane flow $v_z = 0 \Rightarrow$

$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) = 0 \Rightarrow r \cdot v_r = \text{constant}$

At $r=0$ $r \cdot v_r = 0 = \text{constant}$

$v_r = 0$ everywhere

Conservation of momentum

Radial: $$\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r \frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{v_z \frac{\partial v_r}{\partial z}} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left\{ \cancel{\nabla^2 v_r} - \cancel{\frac{v_r}{r^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} \right\}$$

$$\boxed{-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}}$$

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Azimuthal: $\cancel{\frac{\partial V_\theta}{\partial t}} + \cancel{V_r \frac{\partial V_\theta}{\partial r}} + \frac{V_\theta}{r}\cancel{\frac{\partial V_\theta}{\partial \theta}} + \cancel{V_z \frac{\partial V_\theta}{\partial z}} + \cancel{\frac{V_r V_\theta}{r}} = -\frac{1}{\rho r}\cancel{\frac{\partial P}{\partial \theta}} +$ Symmetry

$$+ \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_\theta}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 V_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 V_\theta}{\partial z^2}} + \frac{2}{r^2}\cancel{\frac{\partial V_r}{\partial \theta}} - \frac{V_\theta}{r^2} \right]$$

$$0 = \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_\theta}{\partial r}\right) - \frac{V_\theta}{r^2} \right]$$

$$\frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{1}{\cancel{r}}\cancel{r}\frac{\partial^2 V_\theta}{\partial r^2} - \frac{V_\theta}{r^2} = 0$$

$$\int \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \int \frac{\partial^2 V_\theta}{\partial r^2} = 0$$

$$\frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} = \text{constant}$$

try solution $V_\theta = A \cdot r^{\alpha} \rightarrow \quad A r^{\alpha-1} + A r^{\alpha-1} = \text{constant}$

of the form: $\quad \boxed{\alpha = 1}$

$$V_\theta = A \cdot r$$

At the pail's surface $r = R$ we know that the fluid must be at the same speed as the pail $V_\theta = \Omega R$

so $\boxed{V_\theta = \Omega r}$

If we calculate the vorticity:

$$\vec{\omega} = \left(\frac{1}{r}\cancel{\frac{\partial V_z}{\partial \theta}} - \cancel{\frac{\partial V_\theta}{\partial z}},\ \cancel{\frac{\partial V_r}{\partial z}} - \cancel{\frac{\partial V_z}{\partial r}},\ \frac{1}{r}\frac{\partial}{\partial r}(r V_\theta) - \frac{1}{r}\cancel{\frac{\partial V_r}{\partial \theta}} \right)$$

$$\vec{\omega} = \frac{1}{r}\frac{\partial}{\partial r}(r \Omega r)\vec{k} = \frac{1}{\cancel{r}} 2\Omega \cancel{r} \vec{k} = 2\Omega \vec{k}$$

③

If we calculate the shear stresses:

$$\varepsilon_{r\theta} = \frac{1}{2}\, r \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{2}\,\frac{1}{r}\cancel{\frac{\partial V_r}{\partial \theta}} = \frac{1}{2}\, r \frac{\partial}{\partial r}\left(\frac{\Omega \cancel{r}}{\cancel{r}}\right) = 0$$

$$\varepsilon_{\theta z} = \frac{1}{2}\,\frac{1}{r}\cancel{\frac{\partial V_z}{\partial \theta}} + \frac{1}{2}\cancel{\frac{\partial V_\theta}{\partial z}} = 0$$

$$\varepsilon_{zr} = \frac{1}{2}\cancel{\frac{\partial V_r}{\partial z}} + \frac{1}{2}\cancel{\frac{\partial V_z}{\partial r}} = 0$$

and of course:

$$\varepsilon_{rr} = \frac{\partial V_r}{\partial r} = 0$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} = 0$$

$$\varepsilon_{zz} = \frac{\partial V_z}{\partial z} = 0$$

So although the fluid is flowing, the stresses are zero.

The radial momentum equation turns out to be:

$$\frac{\partial P}{\partial r} = \rho \frac{(\Omega r)^2}{r}$$

and the axial momentum equation is:

$$\frac{\partial P}{\partial z} = -\rho g$$

The pressure distribution is then given by

$$P(r,z) = P_0 - \rho g (z - z_0) + \rho \Omega^2 \frac{(r^2 - r_0^2)}{2}$$

This is different from Bernoulli, as should be for this

$$P + \rho g z + \frac{1}{2}\rho v^2 = P_0 + \rho g z_0 + \frac{1}{2}\rho v_0^2$$

ROTATIONAL FLOW

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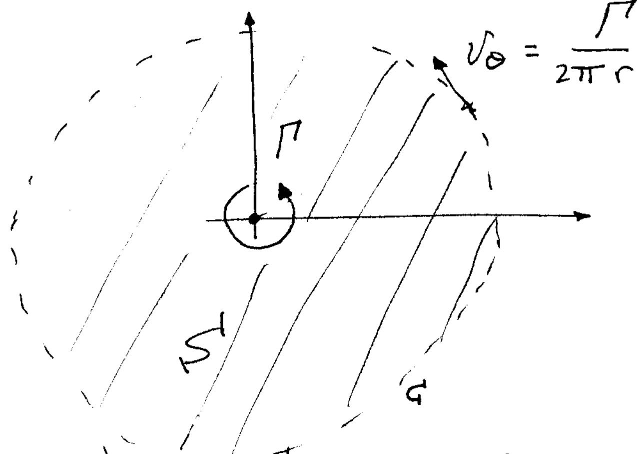
The counter-example is the irrotational vortex. This is an idealized flow by which all the vorticity is concentrated at the origin and the rest of the flow field does not rotate (irrotational).



Γ is the circulation

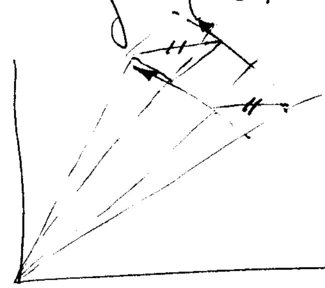
$$\Gamma = \int_S \vec{\omega}\cdot\vec{n}\,dA = \oint_C \vec{v}\cdot d\vec{l}$$

it is an integral measure of the vorticity contained in a region of the flow field.

The shear stresses are:

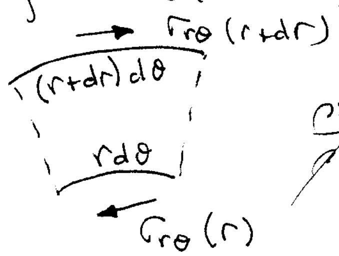
$$\sigma_{r\theta} = 2\mu\,\varepsilon_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial v_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]$$

$$\sigma_{r\theta} = -\mu\frac{\Gamma}{\pi r^2} \quad \text{(not zero)} \Rightarrow$$ Fluid elements are deforming (not rotating)



Because the fluid velocity is inversely proportional to the distance to the origin, fluid elements move without rotation of their diagonals (but deforming its shape).

The resultant force from the shear stress tensor divergence on a fluid element is <u>zero</u>



θ-momentum equation:

$$\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_r v_\theta}{r}} = -\cancel{\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)\right.$$
$$\left. + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}} - \frac{v_\theta}{r^2}\right]$$

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$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r}\right)\right] - \frac{1}{r^2}\frac{\Gamma}{2\pi r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\Gamma}{2\pi r^2}\right)\right] - \frac{\Gamma}{2\pi r^3}$$

$$0 = \frac{1}{r}\left(+\frac{\Gamma}{2\pi}\frac{1}{r^2}\right) - \frac{\Gamma}{2\pi r^3}$$

$$0 = \cancel{\frac{\Gamma}{2\pi r^3}} - \cancel{\frac{\Gamma}{2\pi r^3}} \quad \checkmark$$

In fact the viscous stress tensor divergence can be written, for an incompressible flow, in terms of the vorticity: $\nabla \cdot \bar{\bar{\tau}} = -\mu \nabla \times \vec{\omega}$

For a steady, incompressible, inviscid, irrotational flow, Bernouilli equation applies everywhere: $P + \frac{1}{2}\rho v^2 + \rho g z = P_0 + \frac{1}{2}\rho v_0^2 + \rho g z_0$

$$\frac{1}{2}\rho\frac{\Gamma^2}{4\pi^2 r^2} \qquad \frac{1}{2}\rho\frac{\Gamma^2}{4\pi^2 r_0^2}$$