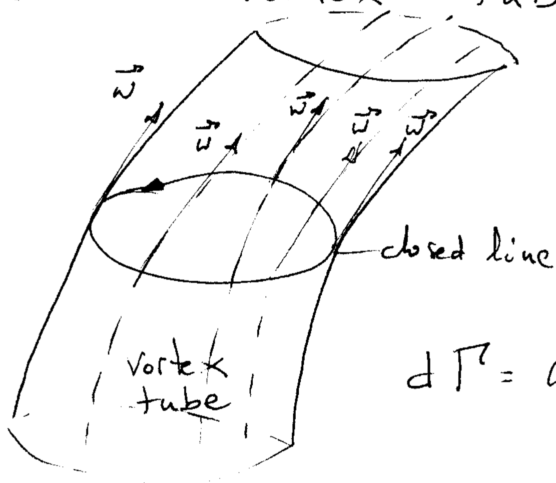
Vortex Lines and Vortex Tubes

Vortex lines are tangent to the vorticity vector everywhere.

Vortex lines passing through a closed line form a surface which is called a vortex tube



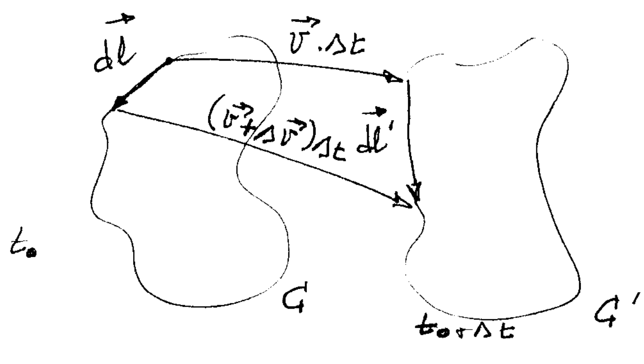
$$d\Gamma = \vec{\omega}\cdot d\vec{A} = \vec{v}\cdot d\vec{l}$$

KELVIN'S CIRCULATION THEOREM

In an inviscid $(\mu \to 0)$, barotropic $(\rho = \rho(P))$ flow with conservative body forces, the circulation around a closed surface moving with the fluid remains constant with time.

$$\frac{D\Gamma}{Dt} = 0$$

$$\Gamma = \oint_C \vec{v}\cdot d\vec{l}$$



$$\Gamma' = \oint_{C'} \vec{v}'\cdot d\vec{l}'$$

$$\frac{D\Gamma}{Dt} = \lim_{\Delta t\to 0}\frac{\Gamma' - \Gamma}{\Delta t} = \frac{\oint_C (\vec{v}' - \vec{v})\cdot d\vec{l}}{\Delta t} + \oint_C \frac{\vec{v}\cdot(d\vec{l}' - d\vec{l})}{\Delta t} + \overset{\text{negligible second order infinitesimal}}{\cancel{\oint \frac{(\vec{v}' - \vec{v})(d\vec{l}' - d\vec{l})}{\Delta t}}}$$

from the vectors in the drawing: $\vec{v}\cdot\Delta t + d\vec{l}' = (\vec{v} + \Delta\vec{v})\Delta t + d\vec{l}$

and therefore $\dfrac{D(d\vec{l})}{Dt} = \dfrac{d\vec{l}' - d\vec{l}}{\Delta t} = d\vec{v}$

Navier-Stokes

$$\frac{D\Gamma}{Dt} = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{l} + \oint_C \vec{v}\cdot\frac{D\,d\vec{l}}{Dt}$$

inviscid

$$\frac{D\Gamma}{Dt} = \oint_C \left(-\frac{1}{\rho}\nabla p + \cancel{\nu\nabla\cdot\bar{\bar{\tau}}'} + \vec{g}\right)\cdot d\vec{l} + \oint_C \vec{v}\cdot d\vec{v}$$

$$\frac{D\Gamma}{Dt} = \cancel{\oint_C -\nabla(P/\rho)\cdot d\vec{l}} + \cancel{\oint_C -\nabla G\cdot d\vec{l}} + \cancel{\oint_C d\left(\tfrac{1}{2}v^2\right)}$$

$$0 \quad \left(\frac{P}{\rho}\right)_2 - \left(\frac{P}{\rho}\right)_1 \qquad 0 \quad -\left[G(2) - G(1)\right]_0 \qquad \tfrac{1}{2}v^2(2) - \tfrac{1}{2}v^2(1)$$

because C' is a closed line 1 and 2 are the same point.

A particular example of this theorem is that irrotational flows remain irrotational, as long as (1) the fluid is inviscid, (2) body forces are conservative, (3) there is a barotropic condition and (4) the reference frame ~~is~~ is not rotating.

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Helmholtz Theorems (under the previous 4 conditions)

1. Vortex lines are material lines (move with the fluid).
2. The strength of a vortex tube is constant along its length. Constant circulation).
3. Vortex tube cannot end on the fluid. It must end on a solid boundary or on itself (closed loop or ring).
4. Strength of a vortex tube remains constant in time

All of these, together with Kelvin's Theorem can be used to interpret physical phenomena and experiments where vortex tubes can be visualized.

Vorticity equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v} - \nabla G$$: Navier Stokes for an incompressible, conservative body forces flow.

Taking the curl of this equation:

$$\frac{\partial}{\partial t}(\nabla\wedge\vec{v}) + \nabla\wedge(\vec{v}\cdot\nabla\vec{v}) = -\nabla\wedge\left(\frac{1}{\rho}\nabla p\right) + \nu\nabla^2(\nabla\wedge\vec{v}) - \nabla\wedge(\nabla G)$$

We need to remember that the curl of a gradient is identically zero, and that $\vec{v}\cdot\nabla\vec{v}$ can be written as $\vec{v}\cdot\nabla\vec{v} = \frac{1}{2}\nabla(v^2) + (\nabla\wedge\vec{v})\wedge\vec{v}$

$$\frac{\partial\vec{\omega}}{\partial t} + \underbrace{\nabla\wedge(\vec{\omega}\wedge\vec{v})}_{\text{triple cross product}} + \frac{1}{2}\cancel{\nabla\wedge(\nabla v^2)} = -\frac{1}{\rho}\cancel{\nabla\wedge\nabla p} - \left(\nabla\frac{1}{\rho}\wedge\nabla p\right) + \nu\nabla^2\vec{\omega} - \cancel{\nabla\wedge(\nabla G)}$$

$$(\vec{v}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{v} - \underbrace{\vec{v}(\nabla\cdot\vec{\omega})}_{0} + \underbrace{\vec{\omega}(\nabla\cdot\vec{v})}_{0}$$

solenoidal fields

Resulting equation is:

$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{v} + \nu\nabla^2\vec{\omega} \underbrace{- \nabla\left(\frac{1}{\rho}\right)\wedge\nabla p}_{\text{Baroclinic torque}}$$

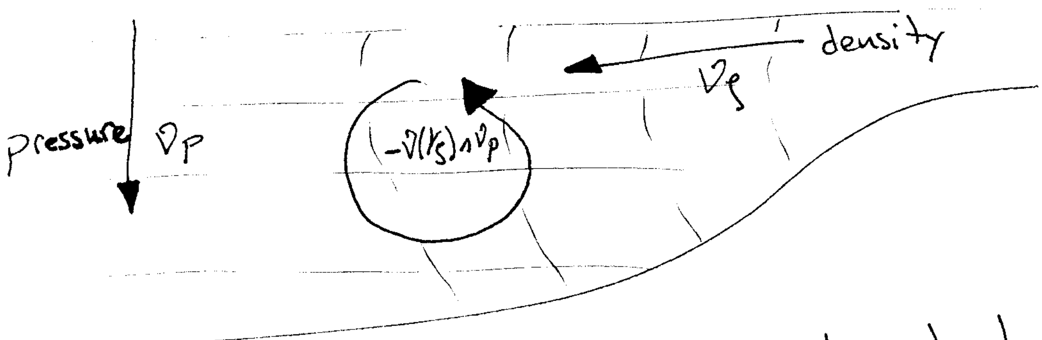
Baroclinic torque
Generates vorticity when the pressure and density fields are not alligned.

Vortex stretching

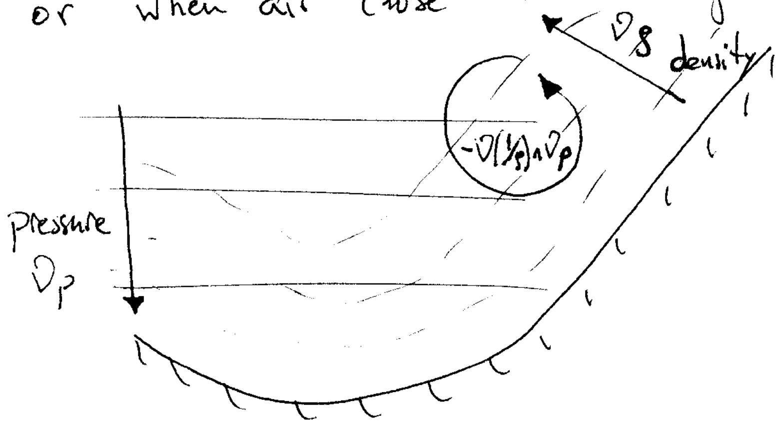
Vortex diffusion

- Baroclinic torque exists when the isopycnals (lines of constant density) are not parallel to the isobars (lines of constant pressure).

This happens, for example when fresh water mixes with salt water in a river estuary:



or when air close to the ground is heated by the sun



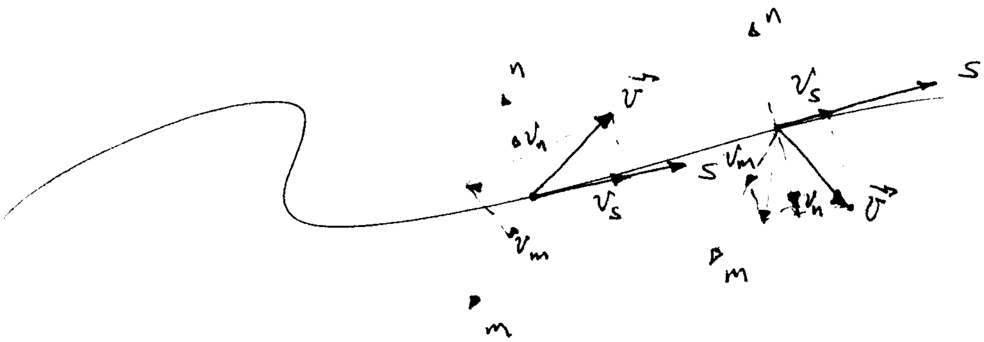
- Vortex Stretching

We can use a reference frame alligned with the vortex line:

s is the coordinate parallel to the local vorticity and n and m are 2 directions (orthogonal) in the perpendicular plane

In this reference frame: $\vec{\omega} = \omega_s \vec{e_s}$ and

$(\vec{\omega} \cdot \nabla)\vec{v} = \omega_s \dfrac{\partial \vec{v}}{\partial s}$ because $\omega_n = 0$ and $\omega_m = 0$.



$$\underbrace{\frac{D\omega_s}{Dt} = \omega_s \frac{\partial v_s}{\partial s}}_{\text{Vortex stretching}} \; ; \quad \underbrace{\frac{D\omega_n}{Dt} = \omega_s \frac{\partial v_n}{\partial s} \; ; \; \frac{D\omega_m}{Dt} = \omega_s \frac{\partial v_m}{\partial s}}_{\text{Vortex tilting}}$$

The magnitude of the vorticity changes by dilation of the fluid element along the direction of the vortex line

Shear in the velocity field $\left(\frac{\partial v_n}{\partial s} \neq 0, \ \frac{\partial v_m}{\partial s} \neq 0\right)$ act as a source of vorticity in the direction perpendicular to the vortex lines ($\omega_n = \omega_m = 0$ initially) and therefore causes the tilting of the vortex lines.

Vorticity is generated somewhere, by baroclinic torque or shear near a boundary, then it is concentrated by vortex stretching or diffused by viscosity or vortex "compression" and also tilted into new directions by shear in the core of the flow.
(away from walls)

Velocity induced by a Vortex filament:

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A concentrated vortex induces a rotational motion in the fluid around it that can be computed from the definition of the vorticity. This is based entirely on kinematic arguments (no dynamics) that ignore the mechanism by which the flow is induced (forces such as pressure)

$$\nabla \wedge \vec{v} = \vec{\omega} \Rightarrow \nabla \wedge \vec{\omega} = \underbrace{\nabla \wedge (\nabla \wedge \vec{v})}_{\text{triple cross product}} = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$$

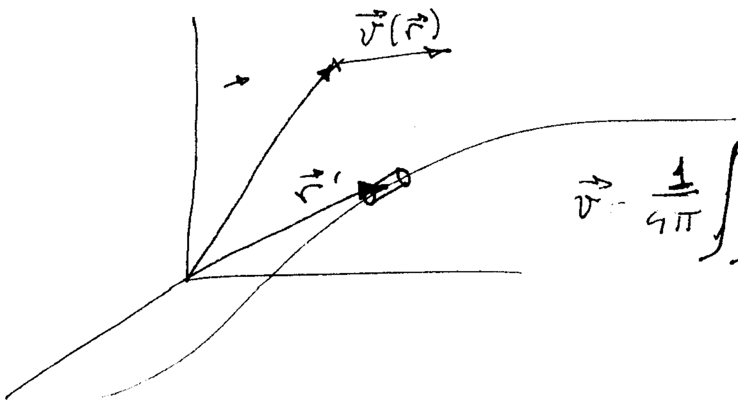
if the flow field is solenoidal ($\nabla \cdot \vec{v} = 0$) for example in an incompressible flow then we have to solve Poisson's equation

$$\nabla^2 \vec{v} = -\nabla \wedge \vec{\omega}$$ where $\vec{\omega}$ is known in a certain domain

The solution for Poisson's equation is:

$$\vec{v}(\vec{r}) = \frac{1}{4\pi} \iiint_{V} \nabla \wedge \vec{\omega} \frac{1}{|\vec{r} - \vec{r}'|} \, dV'$$

$V \leftarrow$ domain where $\nabla \wedge \vec{\omega}$ is non zero



$$\vec{v} = \frac{1}{4\pi} \iiint_{V'} \left[\nabla \wedge \left(\frac{\vec{\omega}}{|\vec{r} - \vec{r}'|} \right) - \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \wedge \vec{\omega} \right] dV'$$

We use the divergence theorem to simplify the first integral by:

$$\iiint_{V'} \nabla \wedge \left(\frac{\vec{\omega}}{|\vec{r}-\vec{r}'|}\right) dV' = -\iint_{S'} \frac{\vec{\omega}}{|\vec{r}-\vec{r}'|} \wedge \vec{dA}' = 0$$

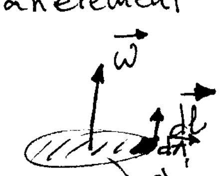
because $\vec{dA}$ is // to $\vec{\omega}$ on the end faces and $\vec{\omega}$ is constant along the filament (kelvin's theorem) so

$$\frac{\vec{\omega}}{|\vec{r}-\vec{r}'|} \wedge \left(\iint_{S'_{sides}} \vec{dA}' \right) = 0$$

and the velocity is:

$$\vec{v} = \frac{1}{4\pi} \iiint + \frac{\nabla |\vec{r}-\vec{r}'| \wedge \vec{\omega}}{|\vec{r}-\vec{r}'|^2} \, dV'$$

the element of volume can be expressed as $dV' = \vec{dA}' \cdot \vec{dl}$ where $\vec{dA}'$ is an element of area from the end faces and $\vec{dl}$ is an element of length parallel to the vortex line.



So the element in volume can be split into integral of area

$$\iint \frac{\nabla |\vec{r}-\vec{r}'|}{|\vec{r}-\vec{r}'|^2} \wedge \vec{\omega} \, \vec{dA}' \, \vec{dl}$$