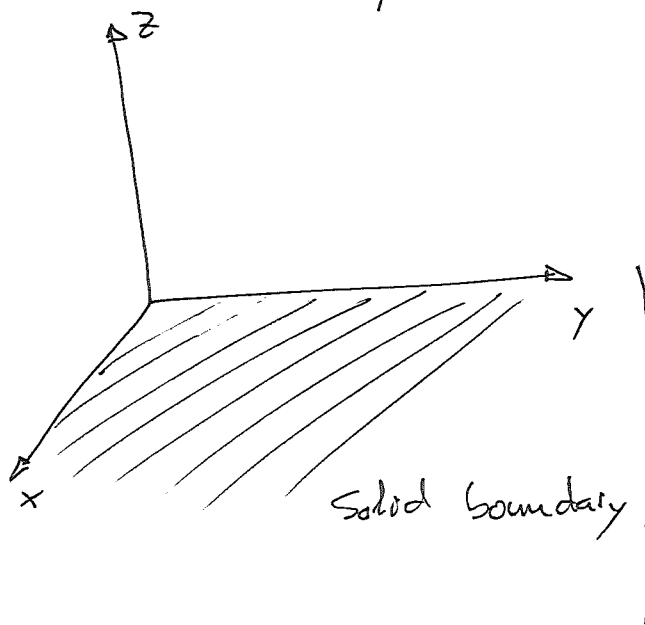


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$$\frac{D\vec{\omega}}{Dt} = -\vec{\omega}(\nabla \cdot \vec{v}) + (\vec{\omega} \cdot \nabla)\vec{v} - \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \left(\frac{1}{\rho} \right) \times \nabla p$$

The only two overall sources of vorticity in the flow are baroclinic torque and viscosity.

In incompressible flows, viscosity is the only source and sink of vorticity: consider incompressible flow near a boundary.



On the boundary ($z=0$)

$$v_x = v_y = v_z = 0$$

and

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} = \frac{\partial v_y}{\partial y} = \frac{\partial v_z}{\partial x} = \frac{\partial v_z}{\partial y} = 0$$

Derivatives along the wall are zero.

Continuity equation shows: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$

So, on the boundary ($z=0$), $\frac{\partial v_z}{\partial z} = 0$

The force due to viscous stress on the boundary is $\vec{f} = \vec{n} \cdot \vec{\tau}' \Big|_{z=0}$ and

$$\vec{n} = (0, 0, 1) = \vec{e}_z \quad \text{so} \quad \vec{f}(z=0) = \left[\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right), \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right), \mu \frac{\partial v_z}{\partial z} \right]_{z=0}$$

We know $\frac{\partial v_x}{\partial x} = \frac{\partial v_z}{\partial y} = \frac{\partial v_z}{\partial z} = 0$ at $z=0$ so

$$\vec{f}(z=0) = \mu \frac{\partial v_x}{\partial z} \Big|_{z=0} \vec{e}_x + \mu \frac{\partial v_y}{\partial z} \Big|_{z=0} \vec{e}_y; \text{ tangent to the wall}$$

If we examine the vorticity on the

$$\text{wall } (z=0): \quad \omega_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = - \frac{\partial v_y}{\partial z} \Big|_{z=0} = - \frac{f_y^{\text{WALL}}}{\mu}$$

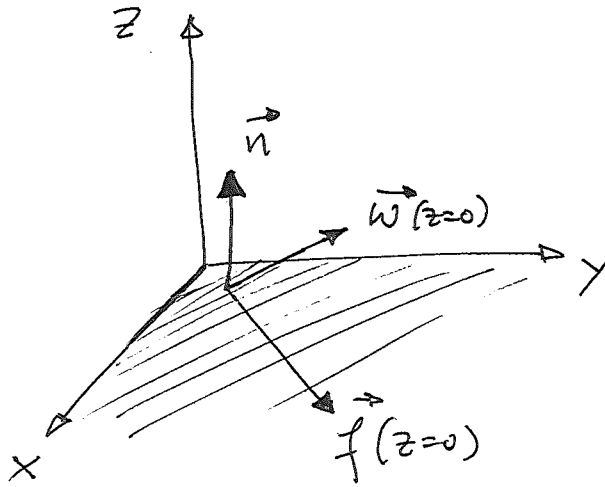
$$\omega_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = \frac{\partial v_x}{\partial z} \Big|_{z=0} = \frac{f_x^{\text{WALL}}}{\mu}$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$$

The vorticity at the boundary is related to the viscous stresses and is also tangent to the wall

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$$\vec{f}(z=0) = \vec{n} \cdot \vec{\tau}' \Big|_{z=0} = -\mu \vec{n} \wedge \vec{\omega} \Big|_{z=0}$$



Vorticity is generated at the wall due to the no-slip condition and then diffuses into the flow domain.

For incompressible flow of a Newtonian fluid:

$$\nabla \cdot \vec{\tau}' = \mu \nabla^2 \vec{v} = -\mu \nabla \wedge \vec{\omega}$$

If we plug this into the Navier-Stokes eq.

evaluated at the boundary: $(\vec{v} \cdot \nabla) v_x = 0$
 $v_y = 0$

$$-\mu \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} \right) \Big|_{z=0} = \frac{\partial P}{\partial x} \Big|_{z=0} + \rho \frac{\partial v_x}{\partial t}$$

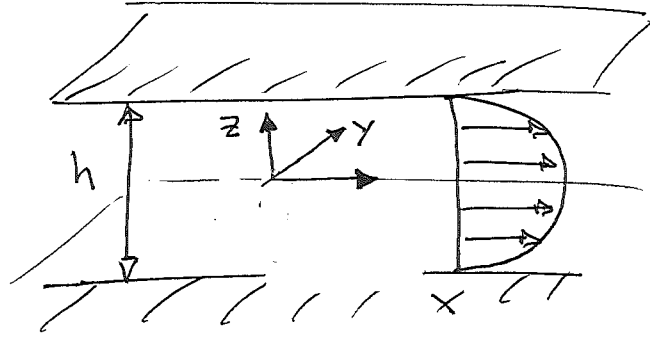
$$-\mu \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} \right) \Big|_{z=0} = \frac{\partial P}{\partial y} \Big|_{z=0} + \rho \frac{\partial v_y}{\partial t}$$

The diffusive flux of vorticity perpendicular to boundary is due to: pressure gradient and fluid accel.

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Examples

Poiseuille Flow



$$U_x(z) = U_{\max} \left[1 - \left(\frac{z}{h} \right)^2 \right]$$

$$U_{\max} = \frac{h^2}{\mu} \frac{\Delta P}{L}$$

$$\frac{dP}{dx} = -\frac{\Delta P}{L}$$

$$\omega_y = \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} = -\frac{U_{\max}}{h} \left(\frac{z}{h} \right)$$

$$\text{In this case: } +\mu \frac{\partial \omega_y}{\partial z} = -\mu \frac{U_{\max}}{h^2} = -\mu \frac{\frac{h^2}{\mu} \frac{\Delta P}{L}}{h^2} =$$

$$= -\frac{\Delta P}{L} = \frac{\partial P}{\partial x} \Big|_{z=0, h/2}$$

Same for the top wall ($z=h/2$) except with opposite sign. The vorticity is generated at the wall and flows (by diffusion, no convective flux) into the fluid domain. Because vorticity has opposite sign at the opposite walls, it cancels at the center where vorticity is 0 $\omega_y(z=0) = 0$