(48)

BUBBLE DYNAMICS

Rayleigh - Plesset equation

LIQUID $\longrightarrow T_\infty, P_\infty$

$R(t)$

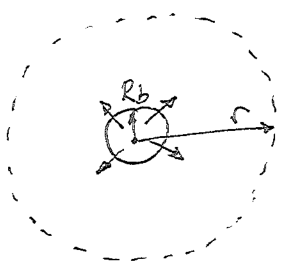
ρ_{liq} = constant (compressibility effects can be studied later as higher order perturbations)

μ_{liq} = constant

The temperature and pressure inside the bubble is assumed to be uniform (small scale of the bubbles supports that assumption)

The boundary condition at the bubble surfaces requires that, assuming negligible mass transfer across the interface,

$v_r\,(r=R_b,t) = \dfrac{dR_b}{dt}$ and continuity imposes



$$\cancel{4\pi}\, R_b^2 \cdot \frac{dR_b}{dt} = \cancel{4\pi}\, r^2\, v_r(r,t)$$

$$v_r(r,t) = \frac{R_b^2 \dfrac{dR_b}{dt}}{r^2}$$

Conservation of momentum for an incompressible, Newtonian fluid is :

$$\frac{\partial v_r}{\partial t} + v_r \cdot \frac{\partial v_r}{\partial r} = -\frac{1}{\rho_{liq}}\frac{\partial P}{\partial r} + \nu_{liq}\left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\, r^2 \frac{\partial v_r}{\partial r}\right) - \frac{2 v_r}{r^2}\right]$$

$\dfrac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\dfrac{1}{\rho}\nabla p + \nu\,\nabla^2\vec{v}$ in spherical coordinates when $v_\theta = v_\phi = \dfrac{\partial}{\partial \theta} = \dfrac{\partial}{\partial \phi} =$

Substituting $v_r = \dfrac{R_b^2 \dfrac{dR_b}{dt}}{r^2}$, $R_b^2 \dfrac{dR_b}{dt} = F(t)$ we get

$$\frac{1}{r^2}\frac{dF}{dt} + \frac{F^2}{r^2}\left(-\frac{2}{r^3}\right) = -\frac{1}{\rho_{liq}}\frac{\partial P}{\partial r} + \cancelto{0}{\nu_{liq}\frac{F(t)}{r^2}\left\{\frac{d}{dr}\left[r^2\left(\frac{-2}{r^3}\right)\right] - \frac{2}{r^2}\right\}}$$

$$\frac{d}{dr}\left(\frac{-2}{r}\right) = \cancel{+\frac{2}{r^2}} - \cancel{\frac{2}{r^2}}$$

$$\boxed{-\frac{1}{\rho_{liq}}\frac{\partial P}{\partial r} = \frac{1}{r^2}\frac{dF}{dt} - \frac{2}{r^5}F^2(t)}$$

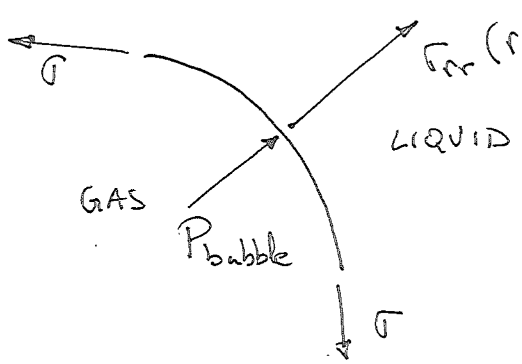
$$\int_r^\infty -\frac{1}{\rho_{liq}}\frac{dP}{dr}\,dr = \int_r^\infty \left[\frac{1}{r'^2}\frac{dF}{dt} - \frac{2}{r'^5}F^2(t)\right]dr'$$

$$\frac{P - P_\infty}{\rho_{liq}} = \left[-\frac{1}{r'}\frac{dF}{dt} + \frac{1}{2r'^4}F^2(t)\right]_r^\infty$$

$$\boxed{\frac{P - P_\infty}{\rho_{liq}} = \frac{1}{r}\frac{dF}{dt} - \frac{1}{2r^4}F^2(t)}$$

Viscous stresses cancel out exactly for ~~axisy~~ spherically symmetric flow.

We have already imposed a kinematic boundary condition at the bubble surface $\left(v_r(r=R_b) = \dfrac{dR_b}{dt}\right)$. We still need to impose dynamic equilibrium on the interface (since the interface has no mass, the forces on it must be balanced).



Equilibrium of normal forces

$$\sigma_{rr} + P_{bubble} - \sigma\left(\frac{1}{R_b} + \frac{1}{R_b}\right) = 0$$

$$\sigma_{rr}\Big|_{r=R} = -P\Big|_{r=R} + 2\mu_{liq}\frac{\partial v_r}{\partial r}\Big|_{r=R} =$$

$$= -P_\infty - \frac{1}{R}\rho_{liq}\frac{dF}{dt} + \frac{1}{2R^4}\rho_{liq}F^2(t) +$$

$$+ 2\mu_{liq}\left[\frac{-2F(t)}{R_b^3}\right]$$

So the boundary condition equation is:

$$P_b - P_\infty = \frac{2\sigma}{R_b} + \frac{1}{R_b}\rho_{liq}\left[2R_b\left(\frac{dR_b}{dt}\right)^2 + R_b^2\frac{d^2R_b}{dt^2}\right] - $$

$$-\frac{1}{2R_b^4}\rho_{liq} R_b^4\left(\frac{dR_b}{dt}\right)^2 + 4\mu_{liq}\frac{R_b^2\frac{dR_b}{dt}}{R_b^3}$$

$$\frac{P_b - P_\infty}{\rho_{liq}} = \frac{2\sigma}{R_b\rho_{liq}} + R_b\frac{d^2R_b}{dt^2} + \frac{3}{2}\left(\frac{dR_b}{dt}\right)^2 + 4\nu_{liq}\frac{1}{R_b}\frac{dR_b}{dt}$$

$P_\infty(t)$ is a forcing term that excites bubble oscillations
$P_b(t)$ is given by the physical processes occuring inside the bubble: polytropic expansion and equilibrium of vapour inside the babble.

The babble is assumed to be filled by a fixed amount of gas that is not transferred to and from the liquid (P_{g_0} at reference radius R_0 and reference temperature T_∞) and vapour from the surrounding liquid that is in thermodynamic equilibrium $\rightarrow P = P_v(T_b)$

$$P_b = P_v(T_b) + P_{g_0}\left(\frac{T_b}{T_\infty}\right)\left(\frac{R_0}{R_b}\right)^3$$

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For inertial bubbles, when thermal effects are negligible, we can write the Rayleigh-Plesset equation as:

$$\frac{P_v - P_\infty}{\rho_{liq}} + \frac{P_{go}}{\rho_{liq}}\left(\frac{R_0}{R_b}\right)^{3K} = \frac{2\sigma/\rho_{liq}}{R_b} + R_b \frac{d^2 R_b}{dt^2} + \frac{3}{2}\left(\frac{dR_b}{dt}\right)^2 + \frac{4\nu_{liq}}{R_b}\frac{dR_b}{dt}$$

The evolution of the gas inside the bubble is assumed to follow a polytropic expasion with exponent K. The two ideal limits of this behaviour are:

- isothermal : $T_b = T_\infty \Rightarrow K = 1$
- isentropic (adiabatic): $\frac{P}{\rho^\gamma} = \text{constant} \Rightarrow P_g = P_{go}\left(\frac{\rho_b}{\rho_o}\right)^\gamma = P_{go}\left(\frac{R_0}{R_b}\right)^{3\gamma}$

  $K = \gamma$ (1.4 for calorically perfect, diatomic gases)

If we assume an oscillatory excitation pressure $P(t) = P_0 - P_A e^{i\omega t}$ which changes sinusoidally with $P_A \ll P_0$ frequency ω and small amplitude, we can solve the equation analytically:

$$R_b(t) = R_0 + R_\epsilon(t) \quad \text{where} \quad R_\epsilon \ll R_0$$

$$\frac{P_v(T_b)}{\rho_{liq}} - \frac{P_0 + P_A}{\rho_{liq}} e^{i\omega t} + \frac{P_{go}}{\rho_{liq}}\left[1 - 3K\frac{R_\epsilon}{R_0} + o\left(\frac{R_\epsilon}{R_0}\right)^2\right] = \frac{2\sigma/\rho_{liq}}{R_0}\left[1 - \frac{R_\epsilon}{R_0} + o\left(\frac{R_\epsilon}{R_0}\right)^2\right]$$

$$+ (R_0 + R_\epsilon)\frac{d^2 R_\epsilon}{dt^2} + \frac{3}{2}\left(\frac{dR_\epsilon}{dt}\right)^2 + \frac{4\nu_{liq}}{R_0}\left(1 - \frac{R_\epsilon}{R_0}\right)\frac{dR_\epsilon}{dt}$$

52 Linearizing the equation:

$$\frac{P_v(T_b)-P_o}{\rho_{liq}} + \frac{P_A}{\rho_{liq}} e^{i\omega t} + \frac{P_{g_o}}{\rho_{liq}} - 3K \frac{P_{g_o}}{\rho_{liq}} \frac{R_\epsilon}{R_o} = \frac{2\sigma/\rho_{liq}}{R_o} - \frac{2\sigma/\rho_{liq} R_\epsilon}{R_o R_o}$$

$$+ R_o \frac{d^2 R_\epsilon}{dt^2} + \frac{4\nu_{liq}}{R_o} \frac{dR_\epsilon}{dt}$$

$$\boxed{P_{g_o} + P_v(T_\infty) = P_o + \frac{2\sigma}{R_o}}$$

$$\frac{P_v(T_b) - P_o + \overbrace{P_o + \frac{2\sigma}{R_o} - P_v(T_\infty)}^{P_{g_o}}}{\rho_{liq}} = 2\frac{\sigma/\rho_{liq}}{R_o} \Rightarrow \text{order zero in } R_\epsilon/R_o$$

Order 1 in R_ϵ/R_o

$$\frac{P_A}{\rho_{liq}} e^{i\omega t} - 3K \frac{P_o + \frac{2\sigma}{R_o} - P_v(T_\infty)}{\rho_{liq}} \frac{R_\epsilon}{R_o} = -\frac{2\sigma/\rho_{liq}}{R_o}\frac{R_\epsilon}{R_o} +$$

$$+ R_o \frac{d^2 R_\epsilon}{dt^2} + \frac{4\nu_{liq}}{R_o}\frac{dR_\epsilon}{dt}$$; which can be

rewritten as

$$\ddot{R}_\epsilon + \frac{4\nu_{liq}}{R_o^2}\dot{R}_\epsilon + \frac{1}{\rho_{liq} R_o^2}\left[3K\left(P_o + \frac{2\sigma}{R_o}\right) - \frac{2\sigma}{R_o}\right] R_\epsilon = \frac{P_A}{R_o \rho_{liq}} e^{i\omega t}$$

~~Which~~ is the equation of a damped harmonic oscillator

with natural frequency $\omega_o^2 = \frac{1}{\rho_{liq} R_o^2}\left[3K\left(P_o + \frac{2\sigma}{R_o}\right) - \frac{2\sigma}{R_o}\right]$

and damping $\xi = \frac{\frac{4\nu_{liq}}{R_o^2}}{\omega_o}$

If surface tension is negligible, the natural frequency of the bubble becomes: $\omega_0^{No\,\Gamma} = \frac{1}{R_0}\sqrt{\frac{3\kappa P_0}{\rho_{liq}}}$

For very small bubbles, where surface tension dominates, the resonant frequency is $\omega_0 = \sqrt{\frac{2\Gamma}{\rho_{liq} R_0^3}(3\kappa-1)}$

Assumptions used in the derivation of the form of the Rayleigh-Plesset equation above:

- single bubble in infinite medium (spherical symmetry)
- conditions inside the bubble are uniform (small acoustic Mach number)
- no body forces (gravity, electromagnetic, etc.)
- bulk (dilational) viscosity ~~affects~~ effects are negligible
- $\rho_{liq} >> \rho_{gas}$ and its compressibility is negligible.
- gas mass inside the bubble is constant (not soluble in the liquid.
- vapour pressure is constant during oscillation.

Of these, the easiest and most productive improvement in the model would be to assume compressibility of the liquid (finite speed of sound in the liquid).

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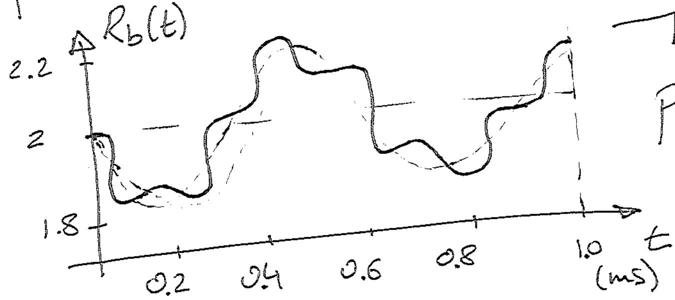
Stable and unstable cavitation (collapse)

Air bubble driven at high amplitude ($P_A = 0.27\ MPa$) but at a frequency far from resonance ($f_{exc} = 10\ kHz$, $f_{res} \approx \frac{3.26\ m/s}{2 \cdot 10^{-3}\ m} = 1.6\ kHz$) responds with low amplitude oscillations.



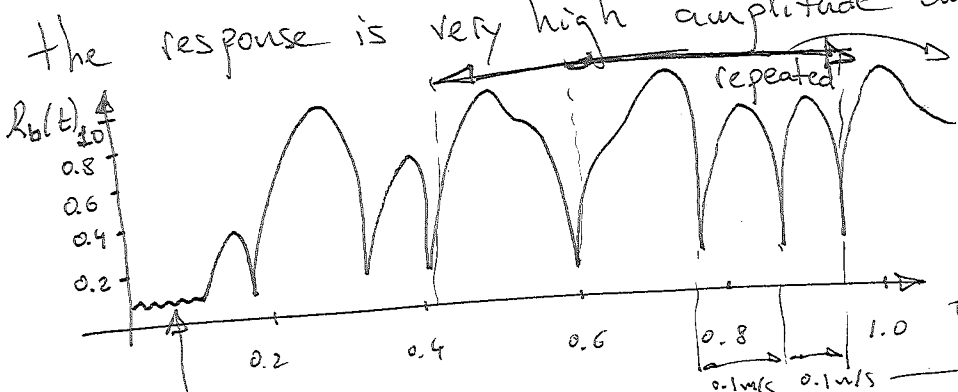
There are 2 distinct frequencies present in the bubble response

1 at 10 kHz (1 peak and valley every $\frac{1}{10}$ milisecond)

and 1 at about 1.6 kHz (1 peak and valley every 0.6 ms).

If we do the same for a 0.1 mm radius bubble, whose resonant frequency is $f \approx \frac{3.26\ m/s}{10^{-4}\ m} \approx 32\ kHz$

the response is very high amplitude and non linear



excitation frequency 10 kHz

initial oscillations at around 70 kHz (period of 0.014 ms), twice the bubble resonant frequency.

Steady state: unit between 0.4 - 0.9 → 2 kHz

These two previous examples are representative of stable cavitation, linear or non-linear.

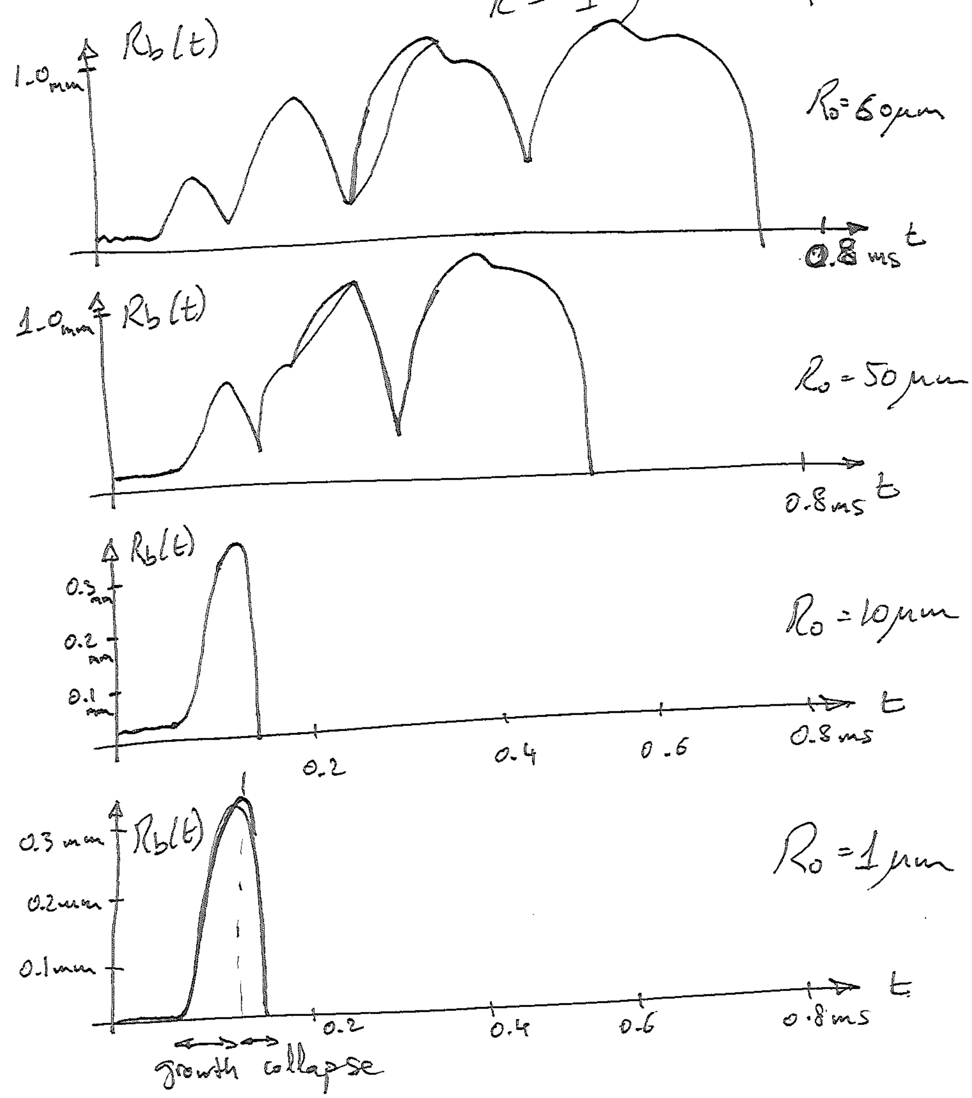
If the bubble was simply a linear oscillator, the behaviour of smaller bubbles would be just to oscillate with smaller amplitude, as their resonant frequencies are further from the excitation $f_{exc} = 10\,kHz$

$f_{res} \approx \dfrac{3.26\,m/s}{R}$ for $R = 60\,\mu m \rightarrow f_{res} = 50\,kHz$

$R = 50\,\mu m \rightarrow f_{res} = 60\,kHz$

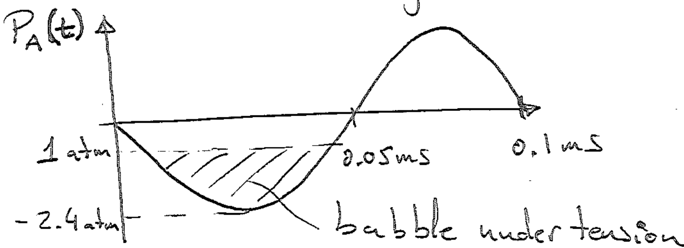
$R = 10\,\mu m \rightarrow f_{res} = 300\,kHz$

$R = 1\,\mu m \rightarrow f_{res} = 3\,MHz$



Because of the significant nonlinearities in the Rayleigh-Plesset equation, the bubble behaviour for these large amplitude excitation differs a lot from the linear oscillator even for this frequency which is far from resonance.

The small bubbles respond to high frequencies. For this "low" frequency excitation the bubbles have a long time under tension to grow to a size much larger than their original size $\frac{R(t)}{R_0} >> 1$.



By the time the pressure becomes compressional, the bubble has grown to a really large diameter, creating a very low pressure inside so that the bubble collapses violently due to its own internal depression as much as the external compression.

Large bubbles have slower response time (lower resonant frequency) and therefore take several cycles to grow to a large size before the violent collapse. Very small bubbles will have their growth impeded by surface tension and may lead to stable cavitation conditions.