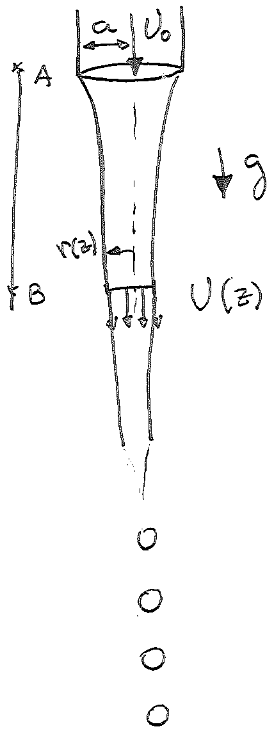
Droplet Breakup



Neglect viscosity

$$P_A + \frac{1}{2}\rho U_0^2 + \rho g z = \frac{1}{2}\rho U^2(z) + P_B$$

$$P_A \approx P_\infty + \frac{\sigma}{a}$$

$$P_B \approx P_\infty + \frac{\sigma}{r}$$

$$\frac{U(z)}{U_0} = \sqrt{1 + \frac{2}{Fr^2}\frac{z}{a} + \frac{2}{We}\left(1 - \frac{a}{r}\right)}$$

$$Fr^2 = \frac{U_0^2}{ga} \qquad We = \frac{\rho U_0^2 a}{\sigma}$$

$$Q = \int_0^r 2\pi r(z)\, U(z)\, dr$$

at $z=0$ $\quad Q = \pi a^2 U_0$

at an arbitrary z : $\quad Q = \pi r^2(z)\, U(z)$

$$\frac{r(z)}{a} = \left[\frac{U_0}{U(z)}\right]^{1/2} = \left[1 + \frac{2}{Fr^2}\frac{z}{a} + \frac{2}{We}\left(1 - \frac{a}{r}\right)\right]^{-1/4}$$

$$\boxed{r^4\left(1 + \frac{2}{Fr^2}\frac{z}{a} + \frac{2}{We}\right) - \frac{2}{We} a r^3 - a^4 = 0}$$

When $We \to \infty$ $\quad \left(1 + \frac{2}{Fr^2}\frac{z}{a}\right) r^4 = a^4$

$$\frac{r}{a} = \left(1 + \frac{2gz}{U_0^2}\right)^{-1/4}$$

$$\frac{U(z)}{U_0} = \left(1 + \frac{2gz}{U_0^2}\right)^{1/2}$$

41

Rayleigh - Plateau instability

$$R = R_0 + \epsilon e^{\omega t + ikz} \qquad \text{where} \quad \epsilon << R_0$$

Based on small perturbations we linearize the equations of motion:

$$\frac{\partial v_r'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} \qquad \text{order } \epsilon \text{ terms}$$

$$\frac{\partial v_z'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$\frac{\partial v_r'}{\partial r} + \frac{v_r'}{r} + \frac{\partial v_z'}{\partial z} = 0$$

Assuming harmonic response of v_r', v_z', p' to the harmonic perturbations, we get:

$$v_r' = V(r)\, e^{\omega t + ikz}$$
$$v_z' = W(r)\, e^{\omega t + ikz}$$
$$p' = P(r)\, e^{\omega t + ikz}$$

$$\left.\begin{aligned} \omega V &= -\frac{1}{\rho}\frac{dP}{dr} \qquad (1) \\ \omega W &= -\frac{ik}{\rho} P \qquad (2) \\ \frac{dV}{dr} + \frac{V}{r} &+ ikW = 0 \qquad (3) \end{aligned}\right\}$$

taking derivatives of (2) and (3) with respect to r and substituting into (1) we get one equation for one unknown $V(r)$

$$\left.\begin{aligned} ik\frac{dW}{dr} &= -\frac{d^2V}{dr^2} + \frac{V}{r^2} - \frac{1}{r}\frac{dV}{dr} \\ ik\frac{dW}{dr} &= \frac{k^2}{\rho\omega}\frac{dP}{dr} \end{aligned}\right\}$$

$$\cancel{\omega} V = \frac{+\cancel{\omega}}{k^2}\left(+\frac{d^2V}{dr^2} \mp \frac{V}{r^2} + \frac{1}{r}\frac{d}{dr}\right.$$

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \left(\frac{1}{r^2} + k^2\right)V = 0$$

(42)

This is a modified Bessel equation of order 1. Its solution can be written in terms of the mod. Bessel functions of the 1st and 2nd kind $I_1(kr)$ and $K_1(kr)$. Since $K_1(kr) \to \infty$ as $kr \to 0$ the solution has to be in terms of

$$V(r) = c\, I_1(kr)$$

The pressure $P(r)$ comes from the properties of the mod. Bessel function $\frac{dP}{dr} = -\rho w V = -\rho w C I_1(kr)$

$$I_0' = I_1(kr) \Rightarrow P(r) = \frac{-\rho w c}{k} I_0(kr)$$

- At the free surface:

$$\frac{\partial R}{\partial t} = \vec{v} \cdot \vec{n} \simeq V_r$$

$$\epsilon w = c\, I_1(kR_0) \Rightarrow c = \frac{\epsilon w}{I_1(kR_0)}$$

- Also, the dynamic b.c. is $P_0 + P' = \sigma \nabla \cdot \vec{n}$

the curvature $\nabla \cdot \vec{n} = \frac{1}{R_1} + \frac{1}{R_2}$ where $\frac{1}{R_1} = \frac{1}{R_0 + \epsilon e^{wt+ikz}} \simeq \frac{1}{R_0}\left(1 - \frac{\epsilon}{R_0} e^{wt+ikz}\right)$

and $\frac{1}{R_2} = \frac{\partial^2 R}{\partial z^2} = \epsilon k^2 e^{wt+ikz}$

$$P_0 + P' = \sigma\left[\frac{1}{R_0}\left(1 - \frac{\epsilon}{R_0} e^{wt+ikz}\right) + \epsilon k^2 e^{wt+ikz}\right]$$

$$P_0 + P' = \sigma\left[\frac{1}{R_0} - \frac{\epsilon}{R_0^2}\left(1 - k^2 R_0^2\right) e^{wt+ikz}\right]$$

$$P_0 = \sigma \frac{1}{R_0} \quad ; \quad P' = -\frac{\epsilon}{R_0^2} \sigma \left(1 - k^2 R_0^2\right) e^{wt+ikz}$$

(43) At $r = R_0$

$$p' = P(R_0)\, e^{i\omega t + ikz} = -\frac{\epsilon}{R_0^2}\sigma\left(1-k^2R_0^2\right)e^{\omega t + ikz}$$

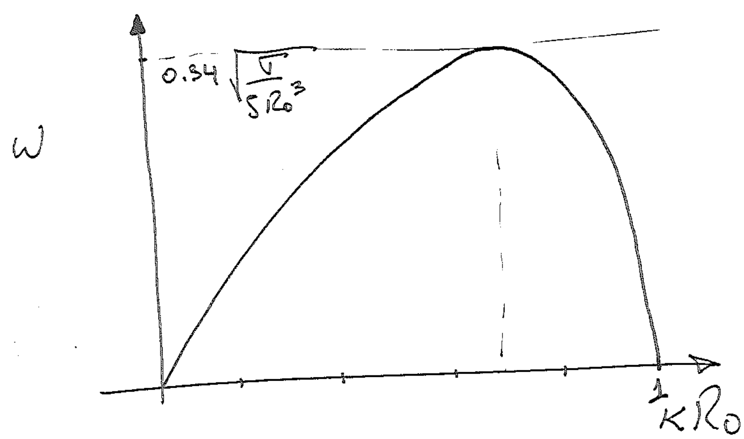
$$-\frac{\omega \rho C}{k} I_0(kR_0) = -\frac{\epsilon}{R_0^2}\sigma\left(1-k^2R_0^2\right)$$

$$C = \frac{\epsilon\omega}{I_1(kR_0)}$$

$$+\frac{\omega\rho\,\epsilon\,\omega}{k\, I_1(kR_0)} I_0(kR_0) = +\frac{\epsilon}{R_0^2}\sigma\left(1-k^2R_0^2\right)$$

$$\omega^2 = \frac{\sigma}{\rho R_0^2}\, k\left(1-k^2R_0^2\right)\frac{I_1(kR_0)}{I_0(kR_0)}$$

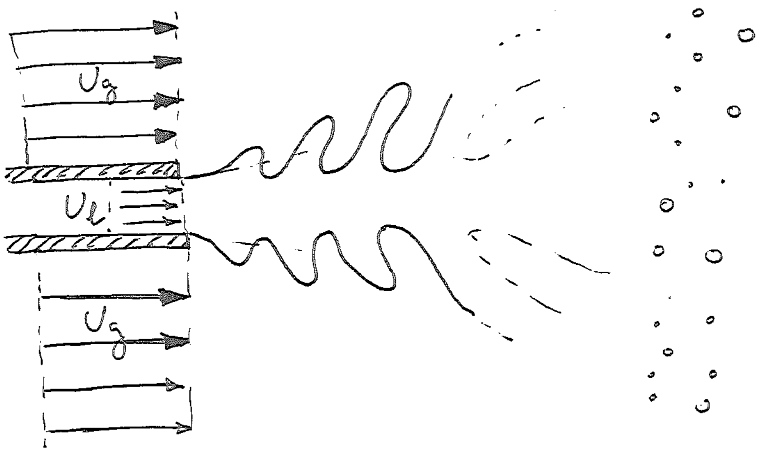
unstable modes (positive and real values of ω) are possible only if $1-k^2R_0^2 \geq 0$ so $k^2R_0^2 \leq 1$



$$\frac{\partial \omega}{\partial k} = 0 \Rightarrow \omega = \omega_{max} \text{ at } k = k_{max}$$

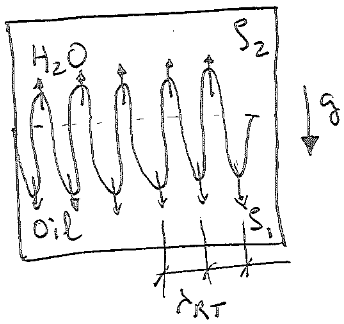
$$0 = \frac{\sigma}{\rho R_0^2}\left(1-3k^2R_0^2\right)\frac{I_1(kR_0)}{I_0(kR_0)} + \frac{\sigma}{\rho R_0^3}\, k\left(1-k^2R_0^2\right)\left(\frac{I_1'}{I_0} - \frac{I_1 \ldots}{I_0^2 \ldots}\right)$$

KELVIN HELMHOLTZ INSTABILITY



RAYLEIGH-TAYLOR INSTABILITY

Heavy fluid over light fluid is always unstable due to the acceleration of gravity (Taylor, 1950).



Rayleigh found that the disturbances of a flat interface grow in time as

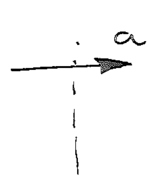
$$e^{nt} \quad \text{where } n = \sqrt{\frac{Kg(\rho_2-\rho_1)}{\rho_2+\rho_1}}$$

$$K = \frac{2\pi}{\ell}$$

This result is unphysical since the growth rate is larger the larger K gets, that is the shorter ℓ is. This can not happen as the solution would go to infinity for arbitrary short times for arbitrarily short lengthscale of the perturbation (micron-nano-molecular-atomic ...)

This result is due to the neglect of surface tension and viscosity which become dominant stabilizing effects for very long wavenumbers (very short wavelengths).

Taylor extended the analysis to an arbitrary accelerat



The stability of this general configuration for a viscous fluid was given by Joseph et al in JFM in 2002:

$$-\left\{1 + \frac{1}{n^2}\left[\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)ak + \frac{\sigma K^3}{\rho_1+\rho_2}\right]\right\}\left(\frac{1}{1+\rho_1/\rho_2}q_1 + \frac{1}{1+\rho_2/\rho_1}q_2 - K\right) - 4K\frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2} +$$

$$+4\frac{k^2}{n}\frac{\mu_1-\mu_2}{\rho_1+\rho_2}\left(\frac{1}{1+\rho_1/\rho_2}q_1-\frac{1}{1+\rho_2/\rho_1}q_2+\frac{\rho_1-\rho_2}{\rho_1+\rho_2}k\right)+4\frac{k^3}{n^2}\left(\frac{\mu_1-\mu_2}{\rho_1+\rho_2}\right)^2(q_1-k)(q_2-k)=0$$

where $q_1=\sqrt{k^2+\frac{n\rho_1}{\mu_1}}$ and $q_2=\sqrt{k^2+\frac{n\rho_2}{\mu_2}}$

For a liquid and a gas, typical of atomization, where $\rho_2 \gg \rho_1$ and $\mu_2 \gg \mu_1$ we obtain

$$-\left[1+\frac{1}{n^2}\left(-ak+\frac{\sigma k^3}{\rho_{liq}}\right)\right]\cancel{(q_1-k)}+4\frac{k^2}{n}\left(\frac{-\mu_{liq}}{\rho_{liq}}\right)\cancel{(q_1-k)}+4\frac{k^3}{n^2}\left(\frac{-\mu_{liq}}{\rho_{liq}}\right)^2\cdot\cancel{(q_1-k)}(q_2-k)=0$$

$$-\left[1+\frac{1}{n^2}\left(-ak+\frac{\sigma k^3}{\rho_{liq}}\right)\right]+\frac{-4k^2}{n}\frac{\mu_{liq}}{\rho_{liq}}+\frac{4k^3}{n^2}\left(\frac{\mu_{liq}}{\rho_{liq}}\right)^2(q_2-k)=0$$

If μ_{liq} is negligible, as is the case in water and most hydrocarbon fuels then:

$$-1+\frac{1}{n^2}\left(-ak+\frac{\sigma k^3}{\rho_{liq}}\right)=0$$

$$n=\sqrt{-ak+\frac{\sigma k^3}{\rho_{liq}}}$$ and

the maximum growth rate is for $\frac{\partial n}{\partial k}=0 \Rightarrow \dfrac{-a+\frac{3\sigma k^2}{\rho_{liq}}}{\sqrt{-ak+\frac{\sigma k^3}{\rho_{liq}}}}=0$

$$\boxed{k_{max}=\sqrt{\frac{a\rho_{liq}}{3\sigma}}}$$

When viscous terms are important μ_{liq} is large and

$\frac{n\rho_{liq}}{\mu_{liq}} \ll 1$ and we get $q_2 - k \approx \frac{n\rho_{liq}}{2\mu_{liq}}$

The dispersion relation then reads:

$$n = \frac{-k^2\mu_{liq}}{\rho_{liq}} \pm \sqrt{\frac{k^4\mu_{liq}^2}{\rho_{liq}^2} - \left(\frac{k^3\sigma}{\rho_{liq}} - ka\right)}$$

The flow is unstable if n is positive so

$$\frac{k^4\mu^2}{\rho^2} - \frac{k^3\sigma}{\rho} + ka > \frac{k^4\mu^2}{\rho^2} \Rightarrow ka > \frac{k^3\sigma}{\rho}$$

but to find the most unstable mode we need to

$$n = \frac{k^2\mu}{\rho}\left[\left(1 + \frac{a\rho^2}{k^3\mu^2} - \frac{\sigma\rho}{k\mu^2}\right)^{1/2} - 1\right] \qquad n = 0 \quad k = \sqrt{\frac{a\rho}{\sigma}}$$

and n is max when $4\frac{\mu^2}{\rho^2}k^3 - \frac{3\sigma}{\rho}k^2 + a = 0$