(14)

Equation of motion for a rigid sphere in a nonuniform flow.

Maxey & Riley Physics of Fl (1983)

$$m_p \frac{d\vec{V}}{dt} = m_f \left(\frac{D\vec{u}}{Dt} - \nu \nabla^2 \vec{u} \right)\Bigg|_{Y(t)} - \frac{1}{2} m_f \frac{d}{dt}\left[\vec{V}(t) - \vec{u}(Y(t),t)\right]$$

$$- 6\pi a \mu \left[\vec{V}(t) - \vec{u}(Y(t),t)\right] + a \int_{-\infty}^{t} \frac{\frac{d}{d\tau}\left[\vec{V}(\tau) - \vec{u}(Y(\tau),\tau)\right]}{\sqrt{\pi\nu(t-\tau)}}\, d\tau$$

$$+ (m_p - m_f)\vec{g}$$

(1), (2), (3), (4), (5)

$\vec{V}$ is the particle velocity

$\vec{u}$ is the fluid velocity

$Y(t)$ is the particle position (center of mass)

$\vec{u}(Y(t),t)$ is the fluid velocity measured at the particle position (hypothetical).

a is the particle radius

1. pressure gradient $\longrightarrow \frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu \nabla^2 \vec{u}$ undisturbed incompressible flow
2. added mass
3. viscous stokes drag
4. Basset history term
5. buoyancy

Term (1) should be written as $\frac{D\vec{u}}{Dt}$ since both the pressure and viscous stresses on the surface of the particle exert a force on the particle.

(15)

The presence of a small rigid spherical particle in a flow $\vec{u}$ $\left(\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu\nabla^2\vec{u}\right)$ creates a perturbed flow $\vec{v}$ such that

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu\nabla^2\vec{v}$$

$$\nabla\cdot\vec{v} = 0$$

$$\vec{v} = \vec{V} + \vec{\Omega}\wedge[\vec{x} - \vec{Y}(t)]$$

at $\|\vec{x}-\vec{Y}\| = a$

and $\vec{v} \to \vec{u}$ as $\|\vec{x}-\vec{Y}\| \to \infty$

if the fluid stress tensor is $\bar{\bar{\sigma}} = -p\,\bar{\bar{I}} + \mu\left(\nabla\vec{v} + \nabla\vec{v}^T\right)$

the equation motion of the sphere is determined by

$$m_p\frac{d\vec{V}}{dt} = m_p\vec{g} + \oint_{S\,\text{sphere}} \bar{\bar{\sigma}}\cdot\vec{n}\,dA$$

$$m_p\frac{d\vec{V}}{dt} = m_p\vec{g} + m_f\left.\frac{D\vec{u}}{Dt}\right|_{Y(t)} + \vec{F}^1$$

undisturbed flow, assuming $\frac{a}{L} \ll 1$ where L is the scale of flow nonuniformities.

$$m_p\frac{d\vec{V}}{dt} = \overset{(5)}{(m_p - m_f)\vec{g}} + \overset{(1)}{m_f\left.\frac{D\vec{u}}{Dt}\right|_{Y(t)}} - \frac{1}{2}m_f\frac{d}{dt}\overset{(2)}{\left\{[\vec{V}(t) - \vec{u}(Y(t),t)] - \frac{1}{10}a^2\nabla^2\vec{u}\Big|_{Y(t)}\right\}} -$$

$$- 6\pi a\mu\overset{(3)}{\left\{[\vec{V}(t) - \vec{u}(Y(t),t)] - \frac{1}{6}a^2\nabla^2\vec{u}\Big|_{Y(t)}\right\}} - 6\pi a^2\mu\int_0^t \frac{\frac{d}{d\tau}\overset{(4)}{\left\{\vec{V} - \vec{u}(Y(t),t) - \frac{1}{6}a^2\nabla^2\vec{u}\Big|_{Y(t)}\right\}}}{\sqrt{\pi\nu(t-\tau)}}$$

$\nabla^2\vec{u}\Big|_{Y(t)}$ modifies the added mass (2), Stokes drag (3) and Basset history (4) term to account for flow curvature.

- Aerosol : $\rho_p/\rho_f \gg 1$. Although fluid inertia is negligible, particle inertia is not.
- Particle oscillating at high frequency : vorticity diffusion is reduced giving rise to enhanced drag through Basset history term.
- Particle falling under gravity starting from rest : initial stages $|\vec{v}-\vec{u}|$ is small and Stokes drag is negligible gravity is balanced by inertia (added mass) and history term. Vorticity shed from the particle surface diffuses away and then Stokes drag takes over.

In terms of the relative velocity $\vec{W} = \vec{V} - \vec{u}(Y(t),t)$

$$\left(m_p + \frac{1}{2} m_f\right) \frac{d\vec{W}}{dt} + 6\pi a^2 \mu \int_0^t \frac{d\vec{W}}{d\tau} \frac{d\tau}{\sqrt{\pi\nu(t-\tau)}} + 6\pi a \mu \vec{W} =$$

$$= - m_p \frac{d\vec{u}}{dt} + m_f \frac{D\vec{u}}{Dt} + (m_p - m_f)\vec{g} + a^3 \pi \mu \nabla^2 \vec{u} + \frac{1}{20} a^2 m_f \frac{d}{dt} \nabla^2 \vec{u}\Big|_{Y(t)}$$

$$+ \pi \mu a^4 \int_0^t \frac{d}{d\tau}\left(\nabla^2 \vec{u}\Big|_{Y(\tau)}\right) \frac{d\tau}{\sqrt{\pi\nu(t-\tau)}}$$

Non-zero Reynolds numbers :

- Oseen
- modified drag due to particle rotation
- Saffman lift due to shear

(17)

To apply this equation of motion to particle in a turbulent flow, we need to consider the different length and velocity scales in turbulence:

- large scales ⇒ L (integral length scale) and u' (velocity rm
- small scales ⇒ η_K (Kolmogorov microscale) and v_K (Kolmogorov velocity)

$\frac{v_K}{u'} = O\left(Re_\lambda^{-1/2}\right)$; $\frac{\eta_K}{L} = O\left(Re_\lambda^{-3/2}\right)$ Re_λ is the Taylor microscale.

For this equation to apply, we require $\frac{a}{\eta_K} \ll 1$ since η_K is smallest scales at which the velocity is not uniform.

Turbulent diffusion is dominated by large scales stirrin so velocities of the order of u' over a long time lead to the statistically asymptotic behaviour.

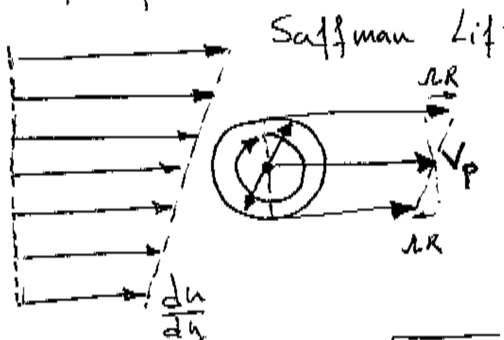
For velocity distribution or spatial distributions, small scale motion has to be taken into account to understand the interaction of particle with turbulent small scales due to inertia

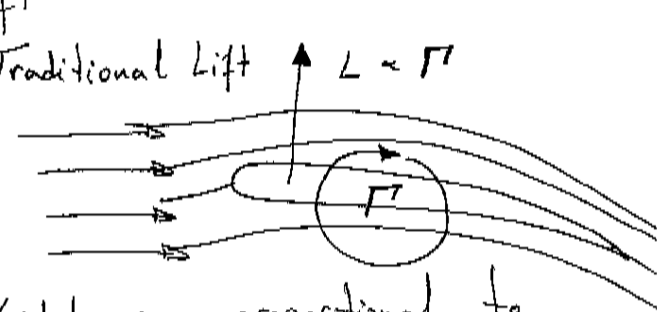
(18)

Other forces acting on a particle in a non-unifor[m] flow are:

- Thermophoresis: due to gradients in temperature
- Photophoresis: due to gradients in incident radiation
- Bjerknes force: due to gradients in pressure (sonophoresis)
- Lift force: - Saffman lift

Saffman Lift | Traditional Lift





Lift is proportional to the circulation

$$L_{Saff} = 1.61\,\mu D\,|\vec{u}-\vec{V}|\sqrt{Re_G}$$

where $Re_G = \dfrac{D^2 \frac{du}{dy}}{\nu}$

$\frac{du}{dy}$ is the flow shear

and $\frac{du}{dy} \cdot D$ is a measure of the velocity difference between top and bottom of the particle.

This can also be expressed in terms of the flow vorticity $\vec{\omega} = \nabla \wedge \vec{u} \Rightarrow \vec{L}_{saffman} = 1.61\, D^2 \sqrt{\frac{\mu \rho}{|\omega|}}\,(\vec{u}-\vec{v}) \wedge \vec{\omega}$

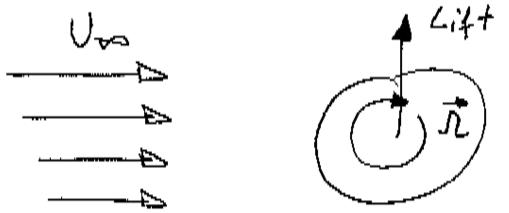
Note that if $\vec{u}-\vec{v}$ is positive, then the lift acts towards the higher velocity, but if $\vec{u}-\vec{v}$ is negative the lift points towards the lower velocity (wall).

Saffman's analysis is based on $\dfrac{|\vec{u}-\vec{V}| \cdot D}{\nu} \ll \dfrac{D^2 \frac{du}{dy}}{\nu} \ll 1$

$Re_{rel} \ll Re_G$

Aerodynamic Lift ≡ Magnus effect

The magnus effect appear when a particle is moving in a fluid with a certain spin imparted on the particle by an outside source other than the fluid

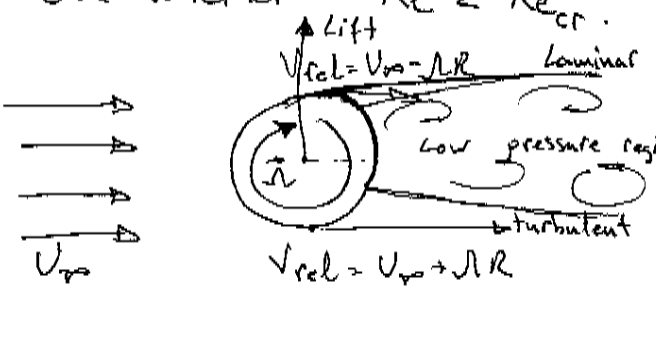


$$\vec{L} = \frac{\pi}{8} D^3 \rho \left[\left(\frac{1}{2} \nabla \wedge \vec{u} - \vec{\Omega} \right) \wedge (\vec{u} - \vec{v}) \right]$$

Where $\frac{1}{2} \nabla \wedge \vec{u}$ is the fluid rotation and $\vec{\Omega}$ is the particle rotation

This force is valid for Reynolds numbers close to unity.

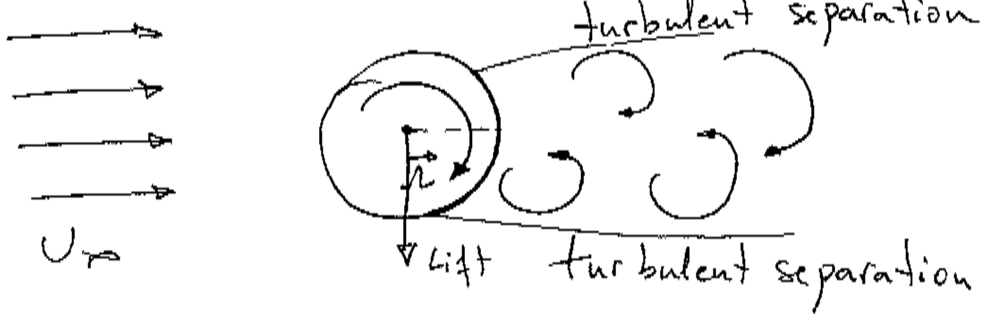
For larger Reynolds numbers, the asymmetric separation of the boundary layer is responsible for the "Magnus" effect. Since the transition to turbulence happens at a critical Reynolds number, and turbulent and laminar boundary layers have very distinct separation behaviour, the Magnus force depends on whether $Re \gtrless Re_{cr}$.



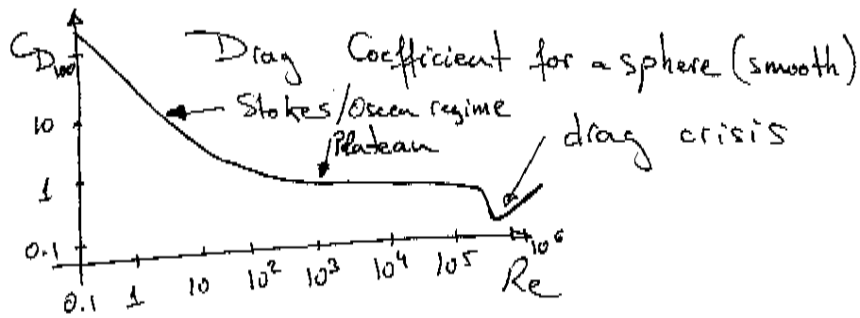
if $Re < Re_{cr}$ but close to it, the b.l. in the lower half of the sphere may become turbulent, therefore delays separation. The upper half is laminar and separates earlier.

$$Re = \frac{V_\infty D}{\nu}$$

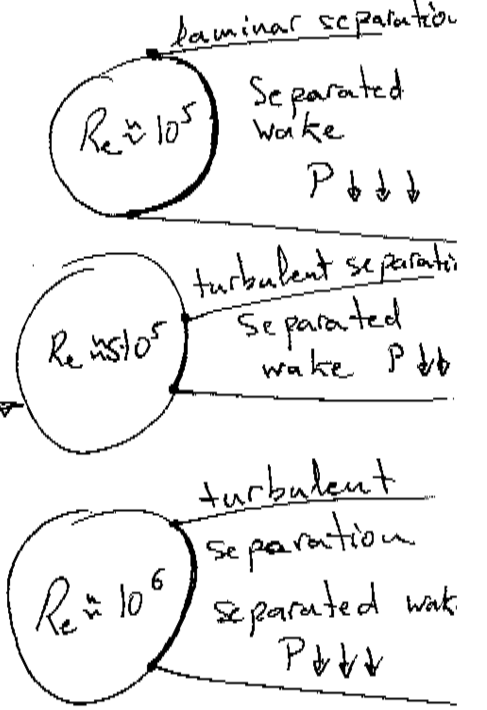
if $Re > R_{cr}$



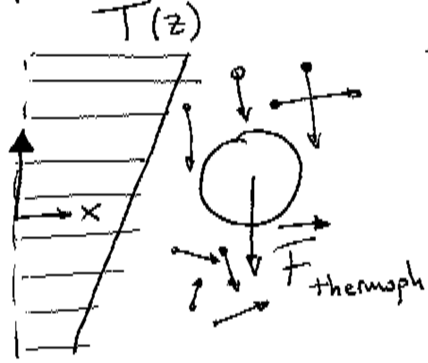
For supercritical conditions, the b.l. is turbulent on both the lower and upper spherical caps and separation occurs earlier on the lower half where the relative velocity is higher



As the Reynolds number increases, the separation point moves upstream, assuming the b.l. is turbulent

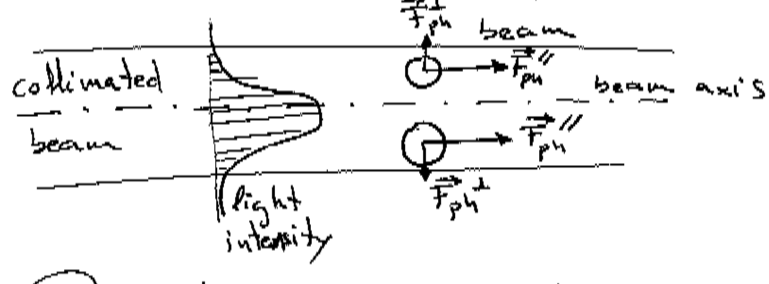


Thermophoresis: a particle immersed in a fluid with a temperature pressure gradient will be subject to a force along the temperature gradient that pushes the particle towards the colder side.

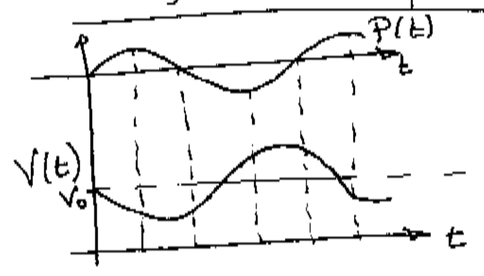


Temperature is a measure of the fluctuating kinetic energy of molecule in a fluid. The larger the temperatu the more energy the molecules have an the more momentum they will exchan with particle through collisions. If these collisions are unbalanced, the particle receives more momentum on one side than the other and there is a resultant force on it.

Photophoresis: same as thermophoresis but related to momentum exchange with photons in a radiation beam.



Bjerknes force: when pressure goes up, volume of the bubble goes down (and viceversa) but there is a phase lag between the pressure and volume oscillations.



$P = P_0 + Re\left[p^* \sin(kx) e^{i\omega t}\right]$; $R = R_0\left[1 + Re(\varphi e^{i\omega t})\right.$

$$Re(\varphi) = \frac{p^*(\omega^2-\omega_n^2)\sin(kx)}{\rho R_0^2\left[(\omega^2-\omega_n^2)^2 + 4\nu\omega/R_0^2\right]}$$

$\langle F \rangle = -\frac{4}{3}\pi R_0^3 \langle (1+\varphi e^{i\omega t}) \ldots$

Particle dynamics. Inertial effects

Equation of motion can be rewritten as:

$$\left(m_p + \frac{1}{2} m_f\right) \frac{d\vec{V}}{dt} = (m_p - m_f)\vec{g} + \left(m_f + \frac{1}{2} m_f\right) \left.\frac{D\vec{u}}{Dt}\right|_{Y(t)} + 6\pi a \mu (\vec{u} - \vec{V})$$

$$+ 6\pi a^2 \mu \int_0^t \frac{\frac{d}{d\tau}(\vec{u} - \vec{V})}{\sqrt{\pi\nu(t-\tau)}} d\tau + m_f C_L (\vec{u} - \vec{V}) \wedge \vec{\omega}$$

$m_p = \frac{4\pi}{3} a^3 \rho_p$ and $m_f = \frac{4\pi}{3} a^3$ so

$$\left(\rho_p + \frac{1}{2}\rho_f\right) \frac{d\vec{V}}{dt} = (\rho_p - \rho_f)\vec{g} + \frac{3}{2}\rho_f \left.\frac{D\vec{u}}{Dt}\right|_{Y(t)} + \frac{9}{2}\frac{\mu}{a^2}(\vec{u} - \vec{V}) +$$

$$+ \frac{9}{2}\frac{\mu}{a} \int_0^t \frac{\frac{d}{d\tau}(\vec{u} - \vec{V})}{\sqrt{\pi\nu(t-\tau)}} d\tau + C_L \rho_f (\vec{u} - \vec{V}) \wedge \vec{\omega}$$

and dividing by ρ_p we get

$$\left(1 + \frac{1}{2}\frac{\rho_f}{\rho_p}\right) \frac{d\vec{V}}{dt} = \left(1 - \rho_f/\rho_p\right)\vec{g} + \frac{3}{2} \rho_f/\rho_p \frac{D\vec{u}}{Dt} + 9/2 \frac{\rho_f}{\rho_p} \frac{\nu_f(\vec{u} - \vec{V})}{a^2} +$$

$$+ \frac{9}{2\sqrt{\pi}} \rho_f/\rho_p \frac{\sqrt{\nu}}{a} \int_0^t \frac{\frac{d}{d\tau}(\vec{u} - \vec{V})}{\sqrt{t-\tau}} d\tau + C_L \frac{\rho_f}{\rho_p} (\vec{u} - \vec{V}) \wedge \vec{\omega}$$

The combination $\frac{2a^2 \rho_p/\rho_f}{9\nu_f}$ is the characteristic response time of the particle τ_p. The term $\frac{\vec{u} - \vec{V}}{\tau_p}$ is usually the main driver of particle motion in a flow.

- When $\frac{\rho_p}{\rho_f} \gg 1$ (solid particles or liquid droplets in a gas)

the equation is approximated by

$$\frac{d\vec{V}}{dt} = \vec{g} + \frac{\vec{u}-\vec{V}}{\tau_p}$$

and $\frac{\vec{g}}{\tau_p}$ can be written as the settling velocity of a particle in a stationary fluid $\vec{V}_{St}$

$$\boxed{\frac{d\vec{V}}{dt} = \frac{\vec{V}_{St} + \vec{u} - \vec{V}}{\tau_p}}$$

the two main sources of particle motion are Drag and gravity.

- when $\rho_p/\rho_f \ll 1$ (gas bubbles in a liquid) the equation can be approximated by

$$\frac{d\vec{V}}{dt} = -2\vec{g} + 3\frac{D\vec{u}}{Dt} + \overbrace{\left(\frac{18}{2}\frac{\rho_f \nu_f}{\rho_p a^2}\right)}^{\frac{1}{\tau_b}}(\vec{u}-\vec{V}) + \underbrace{\sqrt{\frac{9}{\pi}\left(\frac{\rho_f}{\rho_p}\right)^{-1}}\frac{1}{\sqrt{\tau_b}}\int_0^t \frac{\frac{d}{d\tau}(\vec{u}-\vec{V})}{\sqrt{t-\tau}}}_{\text{negligible}}$$

$$\underbrace{+\, C_L(\vec{u}-\vec{V})\wedge\vec{\omega}}_{???}$$

Typically we can use $\frac{d\vec{V}}{dt} = \frac{\vec{V}_{St} + \vec{u} - \vec{V}}{\tau_b} + 3\frac{D\vec{u}}{Dt}$

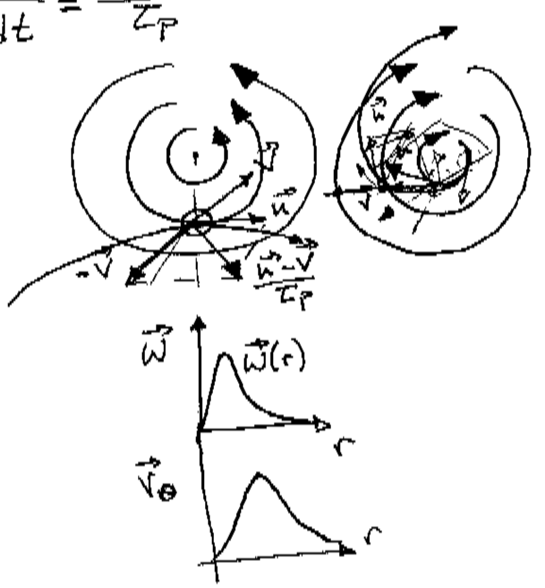
to study the dynamics of bubbles.

$\tau_b = \frac{a^2}{9\nu_f}$; $\vec{V}_{St} = -2\tau_b \cdot \vec{g}$; $3\frac{D\vec{u}}{Dt}$ includes both the effect of added mass (1) and the effect of pressure and viscous stresses on the particle surface due to the non-uniform of the flow (2) (1+2=3).

For heavy particles $\rho_p/\rho_f \gg 1$ result in a centrifuging of particles out of the core of vortices and into the low vorticity / high rate of strain regions of the flow

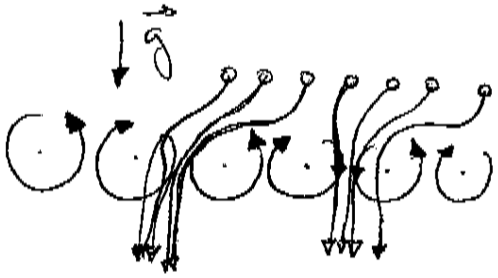
$$\frac{d\vec{v}}{dt} = \frac{\vec{u}-\vec{v}}{\tau_p}$$

The difference between $\vec{u}$ and $\vec{v}$ yield a force that centrifuges the particle out of the core of the vortic "Preferential concentra of particles"



In the presence of gravity, this phenomeno is coupled with the effect of gravitational settlin or raising velocity: crossing trajectories effect.

"Preferential sweeping"

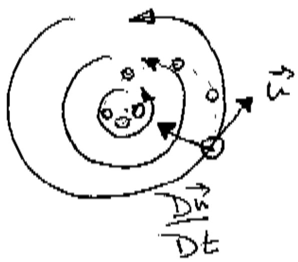


$$\frac{d\vec{v}}{dt} = \frac{\vec{u}-\vec{v}+\vec{V}_{st}}{\tau_p}$$

(25)

For bubbles $\rho_p/\rho_f \ll 1$ The motion can be studied with $\dfrac{d\vec{v}}{dt} = \dfrac{\vec{u} - \vec{v} + \vec{V}_{st}}{\tau_b} + 3\dfrac{D\vec{u}}{Dt}$.

The pressure term $\dfrac{D\vec{u}}{Dt}$ acts to attract bubbles to the core of the vortices where pressur is lower (since they are less dense than the surroundi fluid)



If gravity is acting, bubbles will spent more time on the downward side of eddies than on the upward side, therefore rise velocity will be slower.