4.9. Flow due to a moving body at small Reynolds number

When a body with representative linear dimension d is in steady translational motion, with speed U, through fluid which is otherwise undisturbed, d and U are a representative length and velocity for the flow field as a whole. The inertia forces on the fluid are therefore likely to be of order $\rho U^2/d$ and the viscous forces of order $\mu U/d^2$. The ratio of these two estimates is $\rho dU/\mu$, $= R$, so that when $R \ll 1$ the inertia forces may be negligible. We propose to examine the flow field with this assumption, on the understanding that the solution so obtained must be tested for consistency with the initial assumption. Motion of a body through fluid with a value of R which is small, usually because of the very small size of the body, is a flow problem which is important in a variety of physical contexts, such as the settling of sediment in liquid, and the fall of mist droplets in air. The quantity of greatest practical interest is the drag force exerted by the fluid on the body, since from this the terminal velocity for free fall under the action of gravity can be calculated. The velocity of the body is not always steady in these practical problems, but unless either the body or the ambient fluid is caused to move with an acceleration much greater than U^2/d (as might happen if a sound wave of high frequency passes through the fluid) the above estimate of the relative magnitude of inertia and viscous forces will stand.

The equations to be solved are (4.8.1) and (4.8.2), which we rewrite as

$$\nabla\left(\frac{p-p_0}{\mu}\right) = \nabla^2\mathbf{u} = -\nabla\times\boldsymbol{\omega}, \tag{4.9.1}$$

$$\nabla . \mathbf{u} = 0, \tag{4.9.2}$$

where p_0 is the uniform pressure far from the body. It is a consequence of these equations that

$$\nabla^2 p = 0 \quad \text{and} \quad \nabla^2\boldsymbol{\omega} = 0.$$

We choose a co-ordinate system relative to which the fluid at infinity is stationary. The boundary conditions for a *rigid* body moving with velocity $\mathbf{U}$ are then

$$\left.\begin{aligned} \mathbf{u} = \mathbf{U} \quad &\text{at the body surface,} \\ \mathbf{u}\to 0 \quad \text{and} \quad p-p_0\to 0 \quad &\text{as} \quad |\mathbf{x}|\to\infty. \end{aligned}\right\} \tag{4.9.3}$$

We recognize, from the general result obtained at the end of the preceding section, that not more than one solution of (4.9.1) and (4.9.2) can satisfy the boundary conditions (4.9.3).

We shall make explicit use here of the fact that the equations (4.9.1) and (4.9.2), and the boundary conditions (4.9.3), are linear and homogeneous in $\mathbf{u}$, $(p-p_0)/\mu$ and $\mathbf{U}$. The expressions for $\mathbf{u}$ and $(p-p_0)/\mu$ must therefore be linear and homogeneous in $\mathbf{U}$. (A similar argument was used for irrotational flow in §2.9—see (2.9.23).)

A rigid sphere

The case of a spherical body is important, and is one of the few that are tractable. The flow field due to a rigid sphere in translational motion was first determined by Stokes (1851).

We choose the origin of the co-ordinate system to be at the instantaneous position of the centre of the sphere, which has radius a. The distributions of $\mathbf{u}$ and $(p-p_0)/\mu$ must be symmetrical about the axis passing through the centre of the sphere and parallel to $\mathbf{U}$, and the vector $\mathbf{u}$ lies in a plane through that axis. The differential operators in (4.9.1) and (4.9.2) are independent of the choice of co-ordinate system, so that $(p-p_0)/\mu$ and $\mathbf{u}$ depend on the vector $\mathbf{x}$ and not on any other combination of the components of $\mathbf{x}$. The parameters $\mathbf{U}$ and a complete the list of quantities on which $(p-p_0)/\mu$ and $\mathbf{u}$ can depend (although if the body had been of any shape other than spherical, vectors specifying orientation of the body and scalar shape parameters would have had to be included).

It follows that $(p-p_0)/\mu$ must be of the form $\mathbf{U}.\mathbf{x}F$, where a^2F is a dimensionless function of $\mathbf{x}.\mathbf{x}/a^2\,(=r^2/a^2)$ alone. Since $p-p_0$ satisfies Laplace's equation, and vanishes at infinity, it can be represented as a series of spherical solid harmonics of negative degree in r (see (2.9.19)); and the only term of the series which is compatible with this form is the one of degree -2 (the 'dipole' term). Thus

$$\frac{p-p_0}{\mu}=\frac{C\mathbf{U}.\mathbf{x}}{r^3}, \tag{4.9.4}$$

where C is a constant.

Exactly the same kind of argument applies to the harmonic function $\boldsymbol{\omega}$, which is a vector in the azimuthal direction and must be proportional to $\mathbf{U}\times\mathbf{x}/r^3$. The constant of proportionality is found from (4.9.1) to be C, so that

$$\boldsymbol{\omega}=\frac{C\mathbf{U}\times\mathbf{x}}{r^3}. \tag{4.9.5}$$

The velocity corresponding to this vorticity distribution is most conveniently found in terms of the stream function ψ. With a spherical polar co-ordinate system (and $\theta=0$ in the direction of $\mathbf{U}$), the azimuthal (or ϕ-) component of $\boldsymbol{\omega}$ is defined as

$$\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta},$$

and on replacing u_r, u_θ by the expressions (2.2.14) we find from (4.9.5) that

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)=-\frac{CU\sin^2\theta}{r}.$$

The particular integral for ψ is clearly proportional to $\sin^2\theta$; and the inner boundary condition also requires ψ to depend on θ in this way at $r = a$. We therefore put

$$\psi = U \sin^2 \theta f(r), \tag{4.9.6}$$

which may be seen to be equivalent to a velocity vector of the form

$$\mathbf{u} = \mathbf{U}\left(\frac{1}{r}\frac{df}{dr}\right) + \mathbf{x}\frac{\mathbf{x}.\mathbf{U}}{r^2}\left(\frac{2f}{r^2} - \frac{1}{r}\frac{df}{dr}\right). \tag{4.9.7}$$

The equation for the unknown function f is

$$\frac{d^2f}{dr^2} - \frac{2f}{r^2} = -\frac{C}{r}, \tag{4.9.8}$$

of which the general solution is

$$f(r) = \tfrac{1}{2}Cr + Lr^{-1} + Mr^2. \tag{4.9.9}$$

The terms containing the new constants L and M represent an irrotational motion.

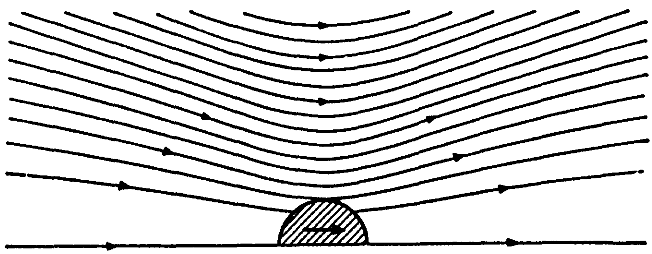

Figure 4.9.1. Streamlines, in an axial plane, for flow due to a moving sphere at $R \ll 1$ (with complete neglect of inertia forces).

Now the outer boundary condition demands that $f/r^2 \to 0$ as $r \to \infty$; and the kinematical condition $u_r = U \cos\theta$ at the surface of the sphere requires $f(a) = \frac{1}{2}a^2$. Hence

$$M = 0, \quad L = \tfrac{1}{2}a^3 - \tfrac{1}{2}Ca^2. \tag{4.9.10}$$

There remains the no-slip condition at the surface of the sphere, viz.

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -U\sin\theta \quad \text{at } r = a,$$

which is satisfied if

$$C = \tfrac{3}{2}a, \quad L = -\tfrac{1}{4}a^3. \tag{4.9.11}$$

The stream function representing the motion is thus

$$\psi = Ur^2 \sin^2\theta\left(\frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}\right). \tag{4.9.12}$$

A sketch of the streamlines is shown in figure 4.9.1. The streamlines are symmetrical about a plane normal to $\mathbf{U}$, as is of course implied by the linearity of $\mathbf{u}$ in $\mathbf{U}$; reversing the direction of $\mathbf{U}$ merely leads to a change of the sign of $\mathbf{u}$ everywhere. It will also be noticed that the disturbance due to the sphere extends to a considerable distance from the sphere, the velocity approaching zero as r^{-1} at large values of r. As a consequence, the presence of an outer rigid boundary, for example in the form of a cylinder with generators parallel to $\mathbf{U}$, can modify the fluid motion appreciably, even when it is at a distance of many diameters from the sphere; likewise the interaction between two moving spheres many diameters apart can be appreciable.

These features of the solution are consequences of the neglect of the inertia term in the equation of motion. The equation for the vorticity, viz. $\nabla^2\boldsymbol{\omega} = 0$, shows that the flow represented by (4.9.12) is effectively due solely to steady molecular diffusion of vorticity to infinity in all directions, the sphere being a source of vorticity as a consequence of the no-slip condition. The term $\partial\boldsymbol{\omega}/\partial t$ which is present in the full equation for $\boldsymbol{\omega}$, and which represents the effect of the continual change in the position of the sphere relative to the axes, has been neglected here, and molecular diffusion spreads the vorticity as far ahead of the sphere as behind it; it is as if the sphere were stationary and acted purely as a source of vorticity. The vorticity distribution shows the decrease as r^{-2} to be expected for the diffusion of each component of $\boldsymbol{\omega}$ from a stationary steady source of dipole character (equal positive and negative quantities of each component of $\boldsymbol{\omega}$ being generated at the surface of the sphere).

It remains for us to verify that the solution found on the assumption that inertia forces can be neglected is actually consistent with that assumption. According to the solution (4.9.12), an estimate of the magnitude of the viscous force $\mu\nabla^2\mathbf{u}$ is $\mu Ua/r^3$. If the sphere velocity is exactly steady, and the rate of change of $\mathbf{u}$ at a fixed point is due simply to the sphere changing its position relative to the point concerned, the operator $\partial/\partial t$ is equivalent to $-\mathbf{U}.\nabla$ and the inertia force is

$$\rho(-\mathbf{U}.\nabla\mathbf{u}+\mathbf{u}.\nabla\mathbf{u}). \tag{4.9.13}$$

For the first of these two terms the order-of-magnitude estimate using (4.9.12) is $\rho U^2a/r^2$, whereas for the second it is $\rho U^2a^2/r^3$. These two terms are of the same order near the sphere, but the first is dominant far from the sphere. Thus the ratio of the order of magnitude of the neglected inertia forces to that of the retained viscous forces is

$$\frac{\rho U^2 a}{r^2}\bigg/\frac{\mu U a}{r^3} = \frac{\rho a U}{\mu}\frac{r}{a} = \tfrac{1}{2}R\frac{r}{a}. \tag{4.9.14}$$

At positions near the sphere our solution is indeed self-consistent when $R \ll 1$, but it seems that the inertia forces corresponding to the solution become comparable with viscous forces at distances from the sphere of order a/R. The solution (4.9.12) is evidently not valid at these large distances from the sphere, although this by itself may not be of consequence since the fluid velocity and the inertia and viscous forces are all small there. We shall in fact see in §4.10 that it is possible to find a velocity distribution which is a valid approximation to the solution of the complete equation of motion everywhere in the fluid when $R \ll 1$, and which coincides with the above solution, to a consistent approximation, when r/a is of order unity.

In order to find the force exerted by the fluid on the sphere, we now evaluate the stress tensor at $r = a$. The i-component of the force per unit area exerted on the sphere at a position denoted by $\mathbf{x} = a\mathbf{n}$ is

$$n_j(\sigma_{ij})_{r=a} = n_j\left\{-p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}_{r=a},$$

and for a velocity of the form (4.9.7) this may be found with a little working to become

$$= \left\{-pn_i + \mu n_i \mathbf{U}.\mathbf{n}\left(-\frac{f''}{r} + \frac{6f'}{r^2} - \frac{10f}{r^3}\right) + \mu U_i\left(\frac{f''}{r} - \frac{2f'}{r^2} + \frac{2f}{r^3}\right)\right\}_{r=a}, \tag{4.9.15}$$

where f' denotes df/dr. Substitution for p and f from (4.9.4) and (4.9.9), and the use of (4.9.10), then gives

$$n_j(\sigma_{ij})_{r=a} = n_i\left\{-p_0 + \frac{3\mu\mathbf{U}.\mathbf{n}}{a}\left(\frac{2C}{a} - 3\right)\right\} + \frac{3\mu U_i}{a}\left(1 - \frac{C}{a}\right), \tag{4.9.16}$$

and, with the value of C required by the no-slip condition,

$$= -p_0 n_i - \frac{3\mu U_i}{2a}. \tag{4.9.17†}$$

It seems that the force per unit area on the sphere due to the motion has the same vectorial value $-3\mu\mathbf{U}/2a$ at all points on the sphere—a striking result, which however is not true of bodies of different shape nor of a sphere with a non-rigid surface. The first term on the right-hand side of (4.9.17) is simply the same uniform normal stress as in the fluid at infinity, and makes no contribution to the total force on the sphere, which is a retarding or drag force parallel to $\mathbf{U}$ of magnitude

$$D = 6\pi a\mu U. \tag{4.9.18}$$

The expression (4.9.18) is usually known as Stokes's law for the resistance to a moving sphere. It is common practice to express the forces exerted on moving bodies by the fluid in terms of a dimensionless coefficient obtained by dividing the force by $\frac{1}{2}\rho U^2$ and by the area of the body projected on to a plane normal to $\mathbf{U}$; thus the drag coefficient is here

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{24}{R} \quad \left(\text{where } R = \frac{2aU\rho}{\mu}\right). \tag{4.9.19}$$

† This result can also be obtained readily from the expression for the stress at a rigid boundary given in the second exercise at the end of §4.1.

It is now a simple matter to calculate the terminal velocity which a sphere would have when falling freely under gravity through fluid, according to Stokes's law. On taking into account the buoyancy force exerted on the sphere (§4.1), we find for the terminal velocity $\mathbf{V}$ of a sphere of density $\bar{\rho}$

$$6\pi a\mu\mathbf{V} = \tfrac{4}{3}\pi a^3(\bar{\rho}-\rho)\,\mathbf{g},$$

that is,

$$\mathbf{V} = \frac{2}{9}\frac{a^2\mathbf{g}}{\nu}\left(\frac{\bar{\rho}}{\rho}-1\right), \qquad (4.9.20)$$

where $\nu = \mu/\rho$. The corresponding value of the Reynolds number for a sphere falling with its terminal velocity is

$$\frac{2aV\rho}{\mu} = \frac{4}{9}\frac{a^3g}{\nu^2}\left(\frac{\bar{\rho}}{\rho}-1\right). \qquad (4.9.21)$$

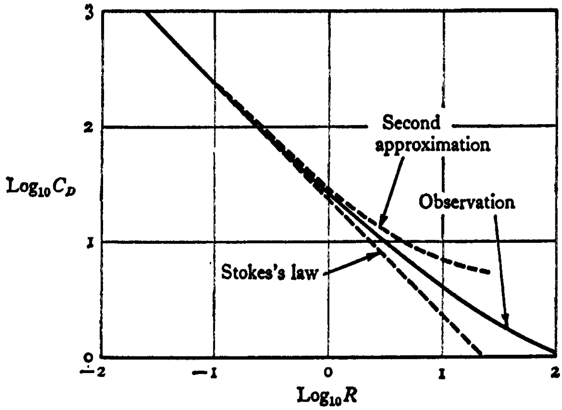

Figure 4.9.2. Comparison of measured values of the drag on a sphere (taken from Castleman 1925) and two theoretical estimates, Stokes's law $C_D = 24/R$, and a second approximation $C_D = 24R^{-1}(1+\frac{3}{16}R)$, where $R = 2a\rho U/\mu$.

For a particle of sand falling through water at 20 °C, we have $\bar{\rho}/\rho \approx 2$ and $\nu = 0{\cdot}010\,\mathrm{cm^2/sec}$, making the Reynolds number $4{\cdot}4\times 10^6 a^3$, a being in centimetres; and for a water droplet (assumed to be rigid) falling through air, we have $\bar{\rho}/\rho \approx 780$ and $\nu = 0{\cdot}15\,\mathrm{cm^2/sec}$, making the Reynolds number $1{\cdot}5\times 10^7 a^3$. The assumption on which neglect of the inertia force was based, namely, that $R \ll 1$, is satisfied in the case of the sand particle in water provided $a \ll 0{\cdot}006$ cm and in the case of the water droplet in air provided $a \ll 0{\cdot}004$ cm. The conditions under which the analysis may be applied are thus restricted to extremely small spheres. However, it seems, from a comparison of the observed and calculated terminal velocities of spheres of known size (see figure 4.9.2), that Stokes's law for the drag is tolerably accurate for most purposes when $R < 1$; and there is no detectable error when $R < 0{\cdot}5$. Thus the theoretical requirement 'small compared with' used above may usually in practice be replaced, so far as the drag force is concerned, by simply 'smaller than'.

It will be noticed from figure 4.9.2 that the curve representing Stokes's law lies below the measured values of the drag and below the other theoretical estimate (which will be referred to in the next section). This was to be expected from the general result established at the end of § 4.8; the velocity field obtained by neglecting inertia forces is accompanied by a smaller total rate of dissipation than that for any other solenoidal velocity distribution with the same value of the velocity vector everywhere on the boundary of the fluid, and hence is accompanied by a smaller rate of working by the sphere against fluid forces at a given speed U.

A spherical drop of a different fluid

In a number of cases of practical interest, the sphere in translational motion at small Reynolds number is itself composed of fluid in which differential motion may occur, and it is desirable to see if this internal circulation affects the drag significantly (Hadamard 1911). We shall suppose that the two fluids are immiscible, and that surface tension at the interface is sufficiently strong to keep the 'drop' approximately spherical against any deforming effect of viscous forces. The condition for this is that γ/a (where γ is the coefficient of surface tension) should be large compared with the normal stress due to the motion, of order $\mu U/a$, that is, that

$$\gamma \gg \mu U; \tag{4.9.22}$$

we shall refer again to this requirement at the end of the section. It will also be assumed that the Reynolds number of the motion within the drop is small compared with unity, like that of the motion outside the drop.

The argument used to determine the velocity and pressure distributions for the case of a rigid sphere can be modified without difficulty. The motions both inside and outside the sphere are axisymmetric and satisfy the equations (4.9.1) and (4.9.2) (although with different values of the viscosity). $\mathbf{u}$ and $p-p_0$ must vanish at infinity, as before, and $\bar{\mathbf{u}}$ and $\bar{p}-\bar{p}_0$ (where the overbar indicates a quantity relating to the internal fluid and its motion) are finite everywhere within the sphere. The common kinematical condition at the interface is

$$\mathbf{n}.\mathbf{u} = \mathbf{n}.\bar{\mathbf{u}} = \mathbf{n}.\mathbf{U} \quad \text{at } r = a. \tag{4.9.23}$$

In place of the no-slip condition at the surface of a rigid sphere there are certain dynamical matching conditions. No relative motion of the two fluids can occur at the interface, and the tangential stress exerted at the interface by the external fluid must be equal and opposite to that exerted by the internal fluid.† No information can be obtained from considerations of the normal stress at the interface, since we have supposed that any discontinuity in the normal stress there which cannot be eliminated by an

† We are assuming here that the only mechanical property of the interface is a uniform surface tension; in practice it appears that contaminant molecules may collect at the interface and give rise to other properties (§ 1.9).

appropriate choice of $\bar{p}_0$ is balanced by surface tension acting at a slightly deformed interface. Thus

$$\mathbf{x} \times \mathbf{u} = \mathbf{x} \times \bar{\mathbf{u}} \quad \text{at } r = a, \tag{4.9.24}$$

$$\epsilon_{mki} n_k n_j (\sigma_{ij} - \bar{\sigma}_{ij}) = 0 \quad \text{at } r = a. \tag{4.9.25}$$

The equations and boundary conditions are linear and homogeneous in $\mathbf{u}, p - p_0, \bar{\mathbf{u}}, \bar{p} - \bar{p}_0$ and $\mathbf{U}$, so that the relations (4.9.4) to (4.9.10) still stand, and are supplemented by analogous relations for the internal motion. $\bar{p}$ satisfies Laplace's equation, like p, and the appropriate solution, analogous to (4.9.4), is

$$(\bar{p} - \bar{p}_0)/\bar{\mu} = \bar{C}\mathbf{U}.\mathbf{x},$$

where $\bar{C}$ is a constant. The stream function and velocity within the sphere have the forms (4.9.6) and (4.9.7), but the internal vorticity is

$$\bar{\boldsymbol{\omega}} = -\tfrac{1}{2}\bar{C}\mathbf{U} \times \mathbf{x}$$

and so the right-hand side of the differential equation for $\bar{f}$, analogous to (4.9.8), is $\frac{1}{2}\bar{C}r^2$. Hence

$$\bar{f}(r) = \tfrac{1}{20}\bar{C}r^4 + \bar{L}r^{-1} + \bar{M}r^2. \tag{4.9.26}$$

The need to avoid a singularity at $r = 0$ and the kinematical condition at $r = a$ require

$$\bar{L} = 0, \quad \bar{M} = \tfrac{1}{2} - \tfrac{1}{20}\bar{C}a^2.$$

The velocity within the sphere is thus

$$\bar{\mathbf{u}} = \mathbf{U} - \tfrac{1}{10}\bar{C}\{\mathbf{U}(a^2 - 2r^2) + \mathbf{x}\mathbf{U}.\mathbf{x}\}. \tag{4.9.27}$$

It remains to determine C and $\bar{C}$ from the dynamical matching conditions. From (4.9.24) we have

$$C - \tfrac{1}{2}a = \tfrac{1}{10}\bar{C}a^3 + a.$$

Only the term containing U_i in the general expression (4.9.15) for the stress across the interface contributes to the tangential component, and matching of this tangential component gives

$$\frac{3\mu}{a^2}(a - C) = \tfrac{3}{10}\bar{\mu}a\bar{C}.$$

Hence

$$C = \tfrac{1}{2}a\frac{2\mu + 3\bar{\mu}}{\mu + \bar{\mu}}, \quad \bar{C} = -\frac{5}{a^2}\frac{\mu}{\mu + \bar{\mu}}. \tag{4.9.28}$$

The resultant force exerted on the interface by the external fluid is now obtained by integrating the force per unit area (4.9.16) over the interface A:

$$\begin{aligned}\int n_j(\sigma_{ij})_{r=a}\,dA &= -4\pi\mu U_i C \\ &= -4\pi a\mu U_i \frac{\mu + \frac{3}{2}\bar{\mu}}{\mu + \bar{\mu}}.\end{aligned} \tag{4.9.29}$$

The terminal velocity $\mathbf{V}$ of a fluid sphere of density $\bar{\rho}$ and viscosity $\bar{\mu}$ moving freely under gravity is thus

$$\mathbf{V} = \frac{1}{3}\frac{a^2\mathbf{g}}{\nu}\left(\frac{\bar{\rho}}{\rho} - 1\right)\frac{\mu + \bar{\mu}}{\mu + \frac{3}{2}\bar{\mu}}. \tag{4.9.30}$$

The case of a rigid sphere is recovered by putting $\bar{\mu}/\mu \to \infty$. The case of a spherical gas bubble moving through liquid corresponds (approximately) to the other extreme, $\bar{\mu}/\mu = 0$, together with $\bar{\rho}/\rho = 0$. The speed of a spherical gas bubble rising steadily under gravity is thus given as $\frac{1}{3}a^2g/\nu$. However, observation of the terminal speed V of very small gas bubbles suggests that the drag is often closer to the value $6\pi a\mu V$ than to the expected value $4\pi a\mu V$; this is believed to be because surface-active impurities present in the liquid collect at the bubble surface, with larger concentration at the rear, thereby setting up a gradient of surface tension which resists the surface movement.†

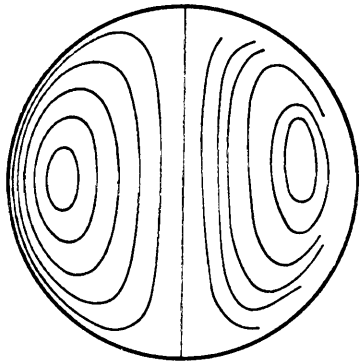

Figure 4.9.3. Comparison of the theoretical and observed pattern of streamlines in a spherical drop of glycerine falling through castor oil (from Spells 1952).

Observations of the general form of the flow inside liquid drops falling under gravity through a second liquid have been made, although measurements of the velocity distribution are difficult. Figure 4.9.3 shows a sketch of the streamlines observed in a spherical liquid drop, relative to axes moving with the drop. The theoretical streamlines corresponding to (4.9.27), relative to these same axes, are lines on which

$$\psi \propto \tfrac{1}{2}Ur^2(a^2-r^2)\sin^2\theta \tag{4.9.31}$$

is constant, and these are also shown; the agreement is satisfactory.

Finally we note an interesting point about the normal component of stress at the surface of the fluid sphere, on which no restrictions have been placed. It will be recalled that the pressure represented in the equations of this section is the modified pressure, and that to obtain the absolute pressure (or a quantity differing from it only by a constant) we should add to the modified pressure a term $\rho\mathbf{g}.\mathbf{x}$ for the flow field outside the sphere and a term $\bar{\rho}\mathbf{g}.\mathbf{x}$ for the flow field within the sphere. The difference between the values of the normal component of the absolute stress at the surface of the

† A general discussion of the effect of adsorbed material at the surface of a small gas bubble rising through liquid will be found in *Physico-chemical Hydrodynamics*, by V. G. Levich (Prentice-Hall, 1962).

sphere as approached from the outer and inner sides is then found from the general expression (4.9.15) to be

$$n_i n_j(\sigma_{ij}-\bar{\sigma}_{ij})_{r=a} = \bar{p}_0 - p_0 - \mathbf{n}.\mathbf{g}a(\rho-\bar{\rho}) + \mathbf{n}.\mathbf{U}\left\{\frac{3\mu}{a^2}(C-2a)+\tfrac{3}{5}a\bar{\mu}\bar{C}\right\}$$

$$= \bar{p}_0 - p_0 - \mathbf{n}.\mathbf{g}a(\rho-\bar{\rho}) - \mathbf{n}.\mathbf{U}\frac{3\mu}{a}\frac{\mu+\frac{3}{2}\bar{\mu}}{\mu+\bar{\mu}}, \qquad (4.9.32)$$

apart from any contribution due to surface tension. The notable feature of the expression (4.9.32) is that when the sphere is moving steadily under gravity, with the velocity of translation then being given by (4.9.30), the normal components of stress differ only by a constant quantity $\bar{p}_0 - p_0$. Thus there is no tendency for the stresses at the interface to deform the sphere and it is not in fact necessary to suppose that the effect of surface tension is so strong as to keep the drop or bubble spherical; surface tension enters only through the relation $\bar{p}_0 - p_0 = 2\gamma/a$ (see (1.9.2)) determining $\bar{p}_0$. Provided the viscosities and densities of the two fluids are such as to make the Reynolds number of the flow so small that inertia forces are negligible, we now see that there is no restriction on the size of the fluid sphere. Air bubbles rising through very viscous liquids such as treacle have been observed to be spherical, even when their radius is so large that the effect of surface tension could not be dominant.

A body of arbitrary shape

Although it is difficult to work out the details of the flow due to a moving body at small Reynolds number for any shape other than spherical,† some general results are available. The following remarks refer only to circumstances in which inertia forces may be neglected completely.

Arguments like those used at the beginning of this section show that, for a body of arbitrary shape in translational motion with velocity **U**, both **u** and $(p-p_0)/\mu$ are linear and homogeneous in **U**. Furthermore, a change of size of the body without change of its shape simply changes the length scale of the whole flow field, so that for a body of given shape $\mathbf{u}/U$ and $(p-p_0)\,d/\mu U$ are (dimensionless) functions of $\mathbf{x}/d$, where d is a representative linear dimension of the body.

Both the tangential and excess normal stresses in the fluid are linear in **U**, so that the resultant (vector) force exerted by the body, given by the integral

$$F_i = -\int \sigma_{ij} n_j \, dA \qquad (4.9.33)$$

taken over the body surface, is proportional to $\mu U d$. The equation (4.9.1) governing flow with negligible inertia forces is equivalent to

$$\partial\sigma_{ij}/\partial x_j = 0,$$

and it follows from an application of the divergence theorem that the integral

† The solution for the case of a rigid ellipsoid is given in *Hydrodynamics*, by H. Lamb, 6th ed. (Cambridge University Press, 1932).

in (4.9.33) has the same value for any surface in the fluid enclosing the body and in particular for a sphere of large radius centred on the origin. Hence

$$F_i = -\int \lim_{r\to\infty} (r\sigma_{ij} x_j)\, d\Omega(\mathbf{x}), \tag{4.9.34}$$

where $\delta\Omega(\mathbf{x})$ is an element of solid angle at the direction of $\mathbf{x}$. This relation shows that in the case of flow due to a moving body exerting a finite force on the fluid $p-p_0$ and the rate-of-strain tensor must both decrease at least as rapidly as r^{-2} as $r \to \infty$.

We know also that $p-p_0$ is a harmonic function and can be represented as a series like (2.9.19). The first non-zero term of this series is evidently of degree -2 in r, so that

$$\frac{p-p_0}{\mu} \sim \frac{P_{ij} U_j d x_i}{r^3} \tag{4.9.35}$$

is the asymptotic form as $r \to \infty$, where P_{ij} is a numerical tensor coefficient dependent only on the body shape. The vorticity $\boldsymbol{\omega}$ also satisfies Laplace's equation, and may be written as a similar series (with allowance for its axial vectorial character). Terms of the same degree in the series for $(p-p_0)/\mu$ and for $\boldsymbol{\omega}$ are related by the governing equation (4.9.1), and it may be seen that if the leading term of the series for $(p-p_0)/\mu$ is $\boldsymbol{\alpha}.\nabla r^{-1}$, that for $\boldsymbol{\omega}$ is $\boldsymbol{\alpha}\times\nabla r^{-1}$. Consequently we have

$$\omega_l \sim \epsilon_{ikl} \frac{P_{ij} U_j d x_k}{r^3}, \tag{4.9.36}$$

as $r \to \infty$. Finally we may obtain the asymptotic form for the velocity, which is determined by (4.9.36) (apart from an irrotational contribution which cannot be of larger magnitude than r^{-3} when the flux of volume across the body surface is zero) and the requirement that $\mathbf{u}$ is solenoidal. We find

$$u_k \sim \tfrac{1}{2} P_{ij} U_j \left(\frac{d}{r}\delta_{ik} + \frac{d}{r^3} x_i x_k\right), \tag{4.9.37}$$

as $r \to \infty$.

It is now possible to relate the coefficient P_{ij} to the force $\mathbf{F}$ by evaluating the stress at a spherical surface of large radius (see (4.9.34)). The working is straight-forward, and leads to the result

$$F_i = 4\pi\mu P_{ij} U_j d. \tag{4.9.38}$$

It appears that the single numerical tensor P_{ij} is sufficient for the specification of the total force on the fluid and the asymptotic expressions for the pressure and velocity, when a body of given shape moves with translational velocity $\mathbf{U}$; and in the case of a spherical body of radius $\tfrac{1}{2}d$ composed of fluid of viscosity $\bar{\mu}$ we know from the preceding calculation that

$$P_{ij} = \tfrac{1}{2}\delta_{ij} \frac{\mu + \frac{3}{2}\bar{\mu}}{\mu + \bar{\mu}}.$$

The flow at large distances from the body has axisymmetry about the direction of the vector $P_{ij} U_j$. Consequently we may represent the flow in this region in terms of a stream function. With spherical polar co-ordinates (r, θ, ϕ) and the axis $\theta = 0$ in the direction of the vector $P_{ij} U_j$—which is also the direction of the force $\mathbf{F}$—we find from (4.9.37) and (4.9.38) that

$$\psi = \frac{F}{8\pi\mu} r \sin^2\theta \tag{4.9.39}$$

in this region, where F is the magnitude of $\mathbf{F}$. Now (4.9.38) shows that $F/\rho\nu^2$ is of the same order as Ud/ν, which has been assumed to be small compared with unity. It is therefore not surprising that we should have recovered the flow field (4.6.18) due to a force of magnitude small compared with $\rho\nu^2$ applied to the fluid at the origin. When a body of arbitrary shape moves through fluid at small Reynolds number, the distant flow field depends only on the resultant force exerted on the fluid and is not affected by the continual change of position of the body.

These general results have a convenient form for application to the case of a small particle, either solid or fluid, falling freely under gravity. If the volume τ and density $\bar{\rho}$ of the particle are known, the distributions of velocity and pressure far from the particle are immediately obtained from the above formulae by putting

$$\mathbf{F} = (\bar{\rho} - \rho)\tau\mathbf{g};$$

details of the shape of the particle are irrelevant, and it presumably also does not matter whether the particle continually turns over and changes its orientation relative to the direction of gravity or whether it moves on a path which is not vertical.

The flow field represented by (4.9.39) is sometimes referred to as being due to the existence of a 'Stokeslet' at the origin.

Exercises

1. Prove that $\mathbf{U}.\mathbf{F}' = \mathbf{U}'.\mathbf{F}$, where $\mathbf{F}$ and $\mathbf{F}'$ are the forces exerted by a body moving at velocities $\mathbf{U}$ and $\mathbf{U}'$ respectively (at small Reynolds number in both cases), and thence that the coefficient P_{ij} in (4.9.38) is a symmetrical tensor.

2. A rigid sphere of radius a is rotating with angular velocity $\mathbf{\Omega}$ in fluid which is at rest at infinity. Show that when $\rho a^2\Omega/\mu \ll 1$ the couple exerted on the fluid by the sphere is $8\pi\mu a^3\mathbf{\Omega}$.

4.10. Oseen's improvement of the equation for flow due to moving bodies at small Reynolds number

It has been seen that, with complete neglect of inertia forces in the flow due to a body of arbitrary shape and linear dimension d moving with speed U, the fluid velocity is of order Ud/r at large values of the distance r from the body. But the first term in the expression (4.9.13) for the inertia force involves a first-order spatial derivative, whereas the viscous force involves a second-order derivative, and it follows that the local inertia force calculated from this solution is in fact comparable in magnitude with the viscous force when r is of order d/R (where $R = \rho U d/\mu$), as seen earlier in the case of a sphere.

This criticism of the use of equation (4.9.1) to represent flow due to bodies moving through fluid of infinite extent which is otherwise undisturbed was made by Oseen (1910), who also showed how it is possible to improve the equation and thereby to remove the inconsistency. Oseen's improvement applies to cases in which the body is moving with steady velocity **U** and in which the flow relative to the body is steady, in which event the local inertia force is as given in (4.9.13), namely,

$$\rho(-\mathbf{U}.\nabla\mathbf{u}+\mathbf{u}.\nabla\mathbf{u}), \tag{4.10.1}$$

where **u** is the fluid velocity relative to a co-ordinate system fixed in the fluid at infinity as before. Since the first of these two terms becomes dominant at large r, and is responsible for inertia forces being comparable with viscous forces at sufficiently large r, Oseen suggested that it, alone of the two contributions to the inertia force, be retained in the equation of motion. The second term, which presents the greater mathematical difficulty in view of its non-linearity in **u**, is again neglected on the assumption that $R \ll 1$; provided $|\mathbf{u}|$ falls off at least as rapidly as r^{-1} as r increases, this second term remains small relative to the viscous force however large r may be. Near the body the two terms in (4.10.1) are of the same order and will both be small compared with the viscous force, provided $R \ll 1$, so that in this region the suggested equation is neither more nor less accurate than (4.9.1).

The Oseen equations for flow due to a moving body at small Reynolds number are therefore

$$\left.\begin{aligned}\rho\frac{\partial\mathbf{u}}{\partial t} = -\rho\mathbf{U}.\nabla\mathbf{u} = -\nabla p+\mu\nabla^2\mathbf{u},\\ \nabla.\mathbf{u}=0,\end{aligned}\right\} \tag{4.10.2}$$

with the boundary conditions, for a rigid body,

$$\mathbf{u}=\mathbf{U} \quad \text{at the surface of the body,}$$

$$\mathbf{u}\to 0 \quad \text{and} \quad p-p_0\to 0 \quad \text{as} \quad r\to\infty.$$

Although these equations are still linear in the dependent variables **u** and p, they are no longer linear in **u**, p and **U**, and are more difficult to solve than (4.9.1) and (4.9.2).

A rigid sphere

The solution of these new equations for the case of a moving sphere is not known in closed form, but an approximate solution which is consistent with the degree of approximation used in the equations themselves has been found (Lamb 1911). In terms of the stream function, this approximate solution, which will simply be quoted here, is

$$\psi = Ua^2\left[-\frac{1}{4}\frac{a}{r}\sin^2\theta+3(1-\cos\theta)\frac{1-\exp\{-\frac{1}{4}R(1+\cos\theta)\,r/a\}}{R}\right] \tag{4.10.3}$$

at the instant at which the centre of the sphere coincides with the origin, where $R = 2aU\rho/\mu$ as before. This expression is readily seen to satisfy the equations (4.10.2) exactly, and it also makes $\mathbf{u} \to 0$ as $r \to \infty$. Near the sphere, where r/a is of order unity and $Rr/a \ll 1$, it becomes

$$\psi = Ua^2 \sin^2\theta \left\{ -\frac{1}{4}\frac{a}{r} + \frac{3}{4}\frac{r}{a} + O\left(R\frac{r}{a}\right)\right\}, \qquad (4.10.4)$$

and therefore coincides with Stokes's solution (4.9.12)—and in particular satisfies the inner boundary condition—with a relative error of order R. This is just the degree of approximation to which (4.10.2) represents the equations of motion, so that (4.10.3) is as accurate a solution of (4.10.2) as is wanted.

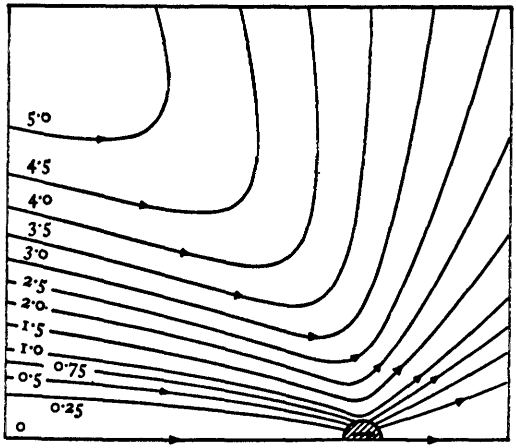

Figure 4.10.1. Streamlines in an axial plane for the outer part of the flow field due to a moving sphere, according to the Oseen equations. ψ is equal to some constant times the numbers shown on the streamlines.

Figure 4.10.1 shows the streamlines corresponding to the solution (4.10.3), with neglect of the first term within square brackets which is significant only near the sphere when $R \ll 1$. The qualitative differences between the Oseen and Stokes solutions in the outer part of the flow field are evident. The streamlines are no longer symmetrical about the plane $\theta = \frac{1}{2}\pi$, as was to be expected from the fact that the governing equation does not remain satisfied after a change in the signs of $\mathbf{u}$ and $\mathbf{U}$. Far from the sphere the flow tends to become radial, as if from a source of fluid at the sphere, except within a 'wake' directly behind the sphere. Analytically, we see from (4.10.3) that when $Rr/a \gg 1$ the flow has different forms according to whether $1 + \cos\theta$ is small compared with unity. At positions where $1 + \cos\theta$ is not small, the stream function becomes

$$\psi \sim Ua^2 \frac{3}{R}(1 - \cos\theta), \qquad (4.10.5)$$

which describes the outward radial flow from a source at the origin emitting

$12\pi a^2 U/R$ units of volume per second. On the other hand, within the wake, where $1+\cos\theta$ is of the same order of magnitude as $4a/rR$ (that is, where $\pi-\theta$ is small and of order $(8a/rR)^{\frac{1}{2}}$), we have

$$\psi \sim Ua^2 \frac{6}{R}\left[1-\exp\left\{-\frac{R}{8}\frac{r}{a}(\pi-\theta)^2\right\}\right], \qquad (4.10.6)$$

which describes a compensating flow towards the sphere, the inflow velocity being $3Ua/2r$ on the axis $\theta = \pi$.

Far from the sphere, the vorticity is zero in the source-flow region and is confined to the wake, which may be regarded as bounded by a paraboloid of revolution on which $(\pi-\theta)^2 r/a$ is of order R^{-1}. Whereas in the Stokes approximation the vorticity diffuses out in all directions from an effectively stationary sphere, here the motion of the sphere is allowed for, as may be seen from the equation for $\boldsymbol{\omega}$ obtained from (4.10.2):

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\mathbf{U}.\nabla\boldsymbol{\omega} = \nu\nabla^2\boldsymbol{\omega}. \qquad (4.10.7)$$

This equation for each component of $\boldsymbol{\omega}$ is of the same form as that satisfied by temperature in a stationary conducting medium through which a steady source (which in this case has a dipole character) of heat is moving with steady velocity $\mathbf{U}$. The vorticity generated at the sphere is left behind as the sphere moves on, in a wake which becomes narrower as R increases.

We may now confirm that the solution (4.10.3) is self-consistent in the way that Stokes's solution was not; that is, we show that the neglected term $\rho\mathbf{u}.\nabla\mathbf{u}$, evaluated by means of (4.10.3), is small compared with any term retained in the equation of motion, when $R \ll 1$. In the region near the sphere, where r/a is of order unity, (4.10.3) reduces to Stokes's solution (with an error of order R), for which $\rho|\mathbf{u}.\nabla\mathbf{u}|$ is already known to be small compared with $\mu|\nabla^2\mathbf{u}|$, the ratio of these terms being of order R. Far from the sphere, in the region where Rr/a is of order unity, which is where (4.10.3) first differs significantly from the Stokes solution, the magnitude of $\mathbf{u}$ as given by (4.10.3) is of order Ua/r, or UR; hence the ratio of the neglected term $\rho|\mathbf{u}.\nabla\mathbf{u}|$ to the retained term $\rho|\mathbf{U}.\nabla\mathbf{u}|$ is of order R and is again small. Still further from the sphere, where $r/a \gg R^{-1}$, $|\mathbf{u}|$ is even smaller by comparison with U.

It appears then that the approximate form of the equation of motion suggested by Oseen has a solution such that the approximation is self-consistent over the whole of the flow field when $R \ll 1$. Near the sphere this solution has the same form as Stokes's solution—and so leads to the same expression, $6\pi a\mu U$, for the resistance experienced by the sphere†—with a

† It will be noticed that the resistance is ρU times the inward flux of volume in the wake far downstream. This relation follows from general considerations of momentum (see §5.12, on wakes), and holds for any body, at any Reynolds number, provided the body moves steadily and leaves behind it a wake of non-zero vorticity whose width increases less rapidly than its length.

relative error of order R, which is the degree of error involved in the replacement of the equation of motion by the Oseen equation. Since (4.10.3) is evidently an approximation to the solution of the complete equations of motion which is valid for $R \ll 1$ over the whole of the flow field, it is natural to consider making (4.10.3) the starting point of a process of successive approximation to the solution of these equations. This has been done (Kaplun and Lagerstrom 1957; Proudman and Pearson 1957), and the second approximation to the drag coefficient has been found to be

$$C_D = \frac{24}{R}(1 + \tfrac{3}{16}R). \tag{4.10.8}$$

(This expression for C_D to order R^0 also follows from the Oseen equations, which at first sight is surprising; the explanation is that the term of order R in the difference between the solution of the Oseen equations for $\mathbf{u}$ and the second approximation to the solution of the complete equations makes zero contribution to the drag for bodies with fore-and-aft symmetry.) As indicated in figure 4.9.2, the formula (4.10.8) agrees with the measured drag for a slightly larger range of Reynolds number than does Stokes's law.

A rigid circular cylinder

The difficulties associated with the use of the equations (4.9.1) and (4.9.2), and the overcoming of these difficulties by the use of the Oseen equations (4.10.2), have been explored for a few other cases of bodies moving steadily through fluid. We shall mention here the case of a circular cylinder of radius a moving with velocity $\mathbf{U}$ normal to its axis, since this case exhibits marked differences, typical of two-dimensional flow at small Reynolds number, from the case of a sphere.

A solution of the equations (4.9.1) and (4.9.2) may be sought in exactly the same way as for a moving sphere, making use of the linearity of the solution in $\mathbf{U}$ and of the dependence on $\mathbf{x}$, $\mathbf{U}$ and a alone. In place of the relations (4.9.4) and (4.9.5) we find

$$\frac{p-p_0}{\mu} = \frac{C\mathbf{U}.\mathbf{x}}{r^2}, \quad \boldsymbol{\omega} = \frac{C\mathbf{U}\times\mathbf{x}}{r^2}, \tag{4.10.9}$$

where C is a constant and (r, θ) are the polar co-ordinates of the two-dimensional vector $\mathbf{x}$. The vorticity may also be expressed in terms of a stream function ψ. The analogue of (4.9.6) is

$$\psi = U \sin\theta f(r), \tag{4.10.10}$$

and the equation satisfied by the function f is

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} = -\frac{C}{r}.$$

The general solution is

$$f(r) = -\tfrac{1}{2}Cr\log r + Lr + Mr^{-1}, \tag{4.10.11}$$

and there arises the difficulty that the particular integral associated with the vorticity distribution gives a divergent velocity at infinity. If for the moment we ignore the outer boundary condition on $\mathbf{u}$, we find that the required conditions at the inner boundary, viz.

$$f/r = 1, \quad df/dr = 1 \quad \text{at } r = a,$$

are satisfied when

$$L = 1 + \tfrac{1}{4}C + \tfrac{1}{2}C \log a, \quad M = -\tfrac{1}{4}a^2C.$$

The velocity distribution is then

$$\mathbf{u} = \mathbf{U} + C\mathbf{U}\left(-\tfrac{1}{2}\log\frac{r}{a} - \frac{1}{4} + \frac{1}{4}\frac{a^2}{r^2}\right) + C\mathbf{x}\frac{\mathbf{U}.\mathbf{x}}{r^2}\left(\frac{1}{2} - \frac{1}{2}\frac{a^2}{r^2}\right). \quad (4.10.12)$$

The normal and tangential stresses at the surface of the cylinder, as derived from the expressions (4.10.9) and (4.10.12), exert a force on the cylinder which is found to be a drag of magnitude

$$D = 2\pi\mu UC \quad (4.10.13)$$

per unit length of the cylinder.

The expressions (4.10.9) and (4.10.12) for $(p-p_0)/\mu$ and $\mathbf{u}$ satisfy the equations (4.9.1) and (4.9.2), the inner boundary condition, and the conditions of linearity in $\mathbf{U}$ and symmetry about $\theta = 0$, but the expression for $\mathbf{u}$ diverges as $\log r$ when r is large and no choice of the remaining arbitrary constant C will make $\mathbf{u} \to 0$ as $r \to \infty$. However, the solution (4.10.12) is not useless. According to (4.10.12), the two contributions to the neglected inertia force (see (4.9.13)) have the following magnitudes when r is large:

$$\left|\rho\frac{\partial \mathbf{u}}{\partial t}\right| \sim \frac{\rho U^2 C}{r}, \quad |\rho\mathbf{u}.\nabla\mathbf{u}| \sim \frac{\rho U^2 C^2}{r}\log\frac{r}{a}. \quad (4.10.14)$$

On the other hand, the retained viscous force has the magnitude

$$|\mu\nabla^2\mathbf{u}| \sim \frac{\mu UC}{r^2}.$$

Both contributions to the inertia force become comparable with the viscous force at sufficiently large distances from the cylinder, the first when r/a is of order R^{-1} (where $R = 2a\rho U/\mu$) and the second when $(Cr/a)\log(r/a)$ is of order R^{-1}. The solution (4.10.12) is thus in any case not a self-consistent approximation to the flow field at large values of r, and its failure to satisfy the outer boundary condition might not therefore be a fatal defect in itself. Evidently some other approximation to the equation of motion at large r is needed, and (4.10.12) must match with the solution of this approximate equation as $r \to \infty$.

Detailed calculation shows that the Oseen approximate form of the equation of motion does in fact have a solution (Lamb 1911) which is self-consistent over the whole field in the sense that the neglected term $\rho\mathbf{u}.\nabla\mathbf{u}$, evaluated according to the solution obtained, proves to be small everywhere by comparison with terms retained in the equation when $R \ll 1$. Near the cylinder this solution for $\mathbf{u}/U$ approximates, with an absolute error of order R, to the form (4.10.12) provided the constant in (4.10.12) is chosen as

$$C = \frac{2}{\log(7\cdot4/R)}. \tag{4.10.15}$$

We note that, with this value of C, the magnitude of $\rho\mathbf{u}.\nabla\mathbf{u}$, according to the the 'inner' solution (4.10.12), does not become comparable with $|\mu\nabla^2\mathbf{u}|$ until r/a is of order R^{-1}, which is also the value of r/a at which the Oseen improvement to equation (4.9.1) is necessary and at which the solution of the Oseen equation has begun to differ from (4.10.12).

The general features of the flow far from the cylinder as obtained from the Oseen equation are similar to those for a sphere, and in particular there is a parabolic wake of finite vorticity behind the cylinder.

Since the solution of the Oseen equation is given approximately by (4.10.12) near the cylinder, with an error of the same order as is involved in the replacement of the equation of motion by the Oseen equation (viz. $O(R)$), the estimate (4.10.13) for the drag is still appropriate. Substituting from (4.10.15), we have for the drag coefficient of unit length of the cylinder

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 2a} = \frac{8\pi}{R\log(7\cdot4/R)}. \tag{4.10.16}$$

It is more difficult to make measurements of the drag on a cylinder at low Reynolds number than for a sphere, owing largely to the unwanted effect of the ends of a cylinder of finite length, but the relation (4.10.16) gives values near $R = 0\cdot5$ which are consistent with observation (see figure 4.12.7).

In some recent research a procedure for obtaining higher-order approximations to the flow past a circular cylinder and to the drag coefficient has been devised.† It appears from these investigations that (4.10.12) (with (4.10.15)) represents the true (non-dimensional) velocity distribution in the neighbourhood of the cylinder with an absolute error of order $(\log R)^{-2}$.

4.11. The viscosity of a dilute suspension of small particles

Mixtures consisting of one material in the form of small particles, either solid, liquid, or gaseous, dispersed randomly throughout another fluid material are quite common in nature and in industry. The term 'suspension' usually refers to a system of small solid particles in liquid, but the nature of the two media is not of particular significance from the dynamical point of view and our use of the word here will include also systems of solid particles in a gas, systems of drops of one liquid dispersed either in another liquid

† For a general account of the procedure, which may be applied to some other problems in fluid dynamics, see *Perturbation Methods in Fluid Mechanics*, by M. D. Van Dyke (Academic Press, 1964).