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FLOW AROUND A SPHERE AT $Re \to 0$

Conservation of mass - Continuity

$\nabla \cdot \vec{v} = 0$ - Incompressible flow

Conservation of Momentum

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

For constant density flows $\rho\vec{g}$ can be written into the reduced pressure gradient

$\rho\vec{g} = -\nabla \rho G$ where $G = -\vec{g}\cdot z\vec{k}$ and

$\mathcal{P} = p + \rho G$

For negligible Reynolds number flows, the inertial term $\rho(\vec{v}\cdot\nabla)\vec{v}$, which scales as $\frac{\rho U_\infty^2}{L}$ is negligible compared to the viscous term $\mu\nabla^2\vec{v}$ which scales as $\frac{\mu U_\infty}{L^2}$

$$Re = \frac{\rho U_\infty^{\cancel{2}}/\cancel{L}}{\mu \cancel{U_\infty}/L^{\cancel{2}}} = \frac{\rho U_\infty L}{\mu} << 1$$ and therefore

assuming that the unsteady term is negligible or scales like the convective term $\left(St = \frac{L}{U_\infty T} \lesssim 1\right)$ the equation simplifies to $\boxed{0 = -\nabla \mathcal{P} + \mu\nabla^2\vec{v}}$

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Taking the curl of the equation we obtain $0 = -\nabla \wedge (\nabla P) + \mu \nabla \wedge \nabla^2 \vec{v}$

and results in $\mu \nabla^2 \vec{\omega} = \nabla \wedge \nabla P = 0$

(it is also true that $\nabla^2 P = 0$ since taking the divergence of the equation $0 = -\nabla . \nabla P + \mu \nabla . \nabla^2 \vec{v}$

$$\nabla^2 P = \mu \nabla^2 (\underbrace{\nabla . \vec{v}}_{0 \text{ continuity}})$$

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The Laplacian of $\vec{\omega} = 0$ gives us a shortcut to solve this equation in terms of a single variable: from the original 4 eq - 4 unknowns we use

Continuity and cons. momentum $(P, \vec{v})$

the spherical symmetry, continuity and the Laplacian being zero to simplify to

1 equation - 1 unknown

The stream function Ψ is defined such that

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}$$

$$v_\theta = \frac{-1}{r\sin\theta}\frac{\partial \Psi}{\partial r}$$

$$\omega_\varphi = -\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial\theta}\right)\right]$$

ω_φ is the only non-zero component by virtue of $\frac{\partial}{\partial\varphi}$ and v_φ being both zero.

$\nabla \wedge \nabla \wedge \vec{\omega} = 0$ becomes $E^4\Psi = 0$ where

$E^4\Psi = E^2(E^2\Psi)$ and $E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$

which is the axisymmetric Laplacian in spherical coordinates.

Boundary conditions are $v_r = v_\theta = 0$ at $r = R$

and

$v_r = V_\infty \cos\theta$
$v_\theta = -V_\infty \sin\theta$ at $r \to \infty$

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$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta$$

$$v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta$$

$$\left.\begin{matrix} v_r \\ v_\theta \end{matrix}\right\} \psi = \frac{1}{2} V_\infty r^2 \sin^2\theta$$

To solve this problem we can assume a solution with separate variables in the form $\psi = f(r)\sin^2\theta$ and we get,

$$E^2\left[\frac{d^2f}{dr^2}\sin^2\theta + \frac{\sin\theta}{r^2} f \frac{d}{d\theta}\left(\frac{1}{\cancel{\sin\theta}} 2\cancel{\sin\theta}\cos\theta\right)\right] = 0$$

$$\frac{d^2}{dr^2}\left[\left(\frac{d^2f}{dr^2} - \frac{2}{r^2}f\right)\sin^2\theta\right] + \frac{\sin\theta}{r^2}\left(\frac{d^2f}{dr^2} - \frac{2f}{r^2}\right)\frac{d}{d\theta}\left(\left(\frac{1}{\sin\theta} 2\sin\cos\theta\right)\right) = 0$$

$$\left(\frac{d^4f}{dr^4} - \frac{12}{r^4}f\right)\sin^2\theta + \left(\frac{8}{r^3}\frac{df}{dr} - \frac{2}{r^2}\frac{d^2f}{dr^2}\right)\sin^2\theta - \frac{2\sin^2\theta}{r^2}\left(\frac{d^2f}{dr^2} - \frac{2f}{r^2}\right) = 0$$

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$$\boxed{r^4 \frac{d^4 f}{dr^4} - 4r^2 \frac{d^2 f}{dr^2} + 8r \frac{df}{dr} - 8f = 0}$$

Euler's equidimensional equation: try solutions of the form $f(r) = r^\alpha$

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3)\, r^\alpha - 4\,\alpha(\alpha-1) + 8(\alpha) - 8 = 0$$

$$\alpha^4 - 6\alpha^3 + 7\alpha^2 + 6\alpha - 8 = 0$$

$$\alpha = -1, 1, 2, 4$$

$$\Psi = U_\infty \sin^2\theta \left(\frac{A}{r} + Br + Cr^2 + Dr^4\right)$$

Boundary conditions: $\underline{\Psi(r\to\infty)} \to \frac{1}{2} U_\infty r^2 \sin^2\theta$

$$D = 0, \quad C = \frac{1}{2}$$

At $r = R$:

$$\frac{\partial \Psi}{\partial r} = 0 \qquad -\frac{A}{R^2} + B + 2CR = 0 \quad \cdot R$$

$$\frac{\partial \Psi}{\partial \theta} = 0 \qquad \frac{A}{R} + BR + CR^2 = 0$$

$$2BR + \frac{3}{2}R^2 = 0 \Rightarrow B = -\frac{3}{4}R$$

$$2\frac{A}{R} - \frac{1}{2}R^2 = 0 \Rightarrow A = \frac{1}{4}R^3$$

$$\boxed{\Psi = U_\infty \sin^2\theta \left(\frac{R^3}{4r} - \frac{3R}{4}r + \frac{1}{2}r^2\right)}$$

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} = \frac{2U_\infty}{\cancel{\ }}\left(\frac{R^3}{4r^3} - \frac{3R}{4r} + \frac{1}{2}\right)\cos\theta$$

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} = U_\infty\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)\sin\theta$$

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The drag on the sphere can be calculated by: $\vec{f} = \vec{n} \cdot \overline{\overline{\sigma}}$ where $\vec{n}$ is the vector normal to the body, in this case $\vec{n} = \vec{e}_r$ since the body is a sphere

$\vec{f}$: force per unit area on the sphere is equal to

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{rr} & \sigma_{\theta r} & \sigma_{\varphi r} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\varphi\theta} \\ \sigma_{r\varphi} & \sigma_{\theta\varphi} & \sigma_{\varphi\varphi} \end{pmatrix} = \sigma_{rr}\vec{e}_r + \sigma_{\theta r}\vec{e}_\theta$$

and $\sigma_{rr} = -p + 2\mu \dfrac{\partial V_r}{\partial r}$

$$\sigma_{\theta r} = \mu \left[r \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta} \right]$$

$$\vec{F} = \int_0^{2\pi}\int_0^{\pi} \vec{f} \underbrace{dA}_{R^2 \sin\theta d\theta d\varphi} = 6\mu\pi R \; U_\infty \underbrace{\left(\cos\theta\, \vec{e}_r - \sin\theta\, \vec{e}_\theta\right)}_{\vec{e}_z}$$

$$\vec{F} = 6\mu\pi R \underbrace{U_\infty \vec{e}_z}_{\vec{V}_\infty}$$

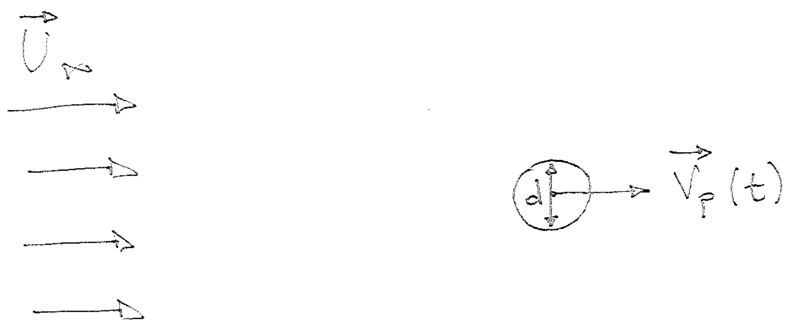
The drag coefficient will therefore be:

$$\frac{|\vec{D}|}{\frac{1}{2}\rho U_\infty^2 \pi R^2} = \frac{6\mu \cancel{\pi} R\, \cancel{U_\infty}}{\frac{1}{2}\rho U_\infty^{\cancel{2}} \cancel{\pi} R^{\cancel{2}}} = \frac{12\mu}{\rho U_\infty R} = \frac{24}{\frac{\rho V_\infty d}{\mu}} = \frac{24}{Re}$$

Reynolds number based on the diameter

Particle response to a uniform velocity carrier flow

Neglect gravity



$$m_p \frac{d\vec{V}_p}{dt} = \vec{Drag} = \frac{1}{2}\rho_f \left|\vec{U}_\infty - \vec{V}_p\right| \left(\vec{U}_\infty - \vec{V}_p\right) \frac{\pi d^2}{4} C_D$$

In the direction of motion:

$$\rho_p \frac{\cancel{\pi} d^{\cancel{3}}}{6} \frac{dV_p}{dt} = \frac{1}{2}\rho_f \left(U_\infty - V_p\right)^2 \frac{\cancel{\pi} \cancel{d^2}}{4} \cdot C_D$$

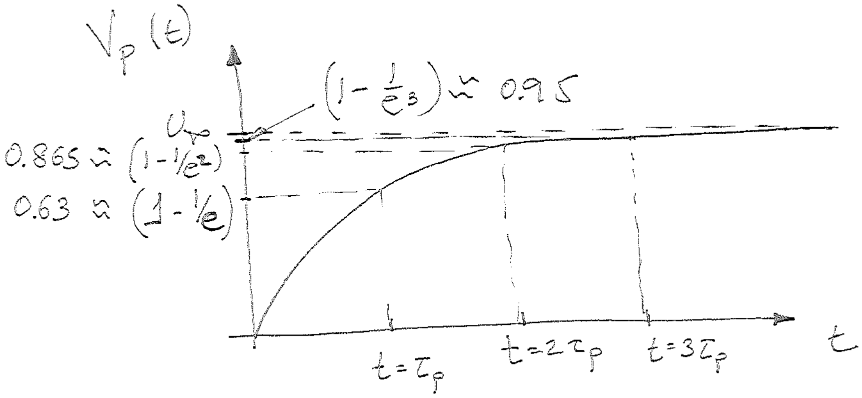
If $Re_p = \dfrac{d\left|\vec{U}_\infty - \vec{V}_p\right|}{\nu_f} \ll 1 \Rightarrow$ Stokes flow gives $C_D = \dfrac{24}{Re_p}$

$$\frac{dV_p}{dt} = \frac{3}{4}\frac{\rho_f}{\rho_p}\frac{\left(U_\infty - V_p\right)^{\cancel{2}}}{d} \cdot \frac{\cancel{24}}{\frac{d\,\cancel{(U_\infty - V_p)}}{\nu_f}} = \frac{18\nu_f}{d^2 \rho_p/\rho_f}\left(U_\infty - V_p\right)$$

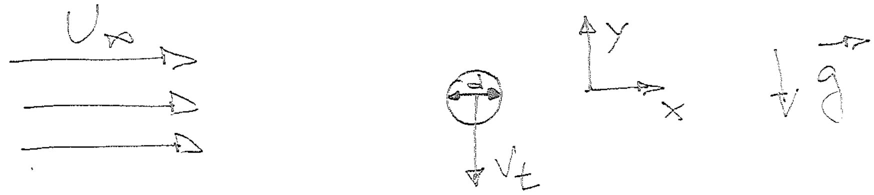
$$\int_0^V \frac{dV_p}{U_\infty - V_p} = \int_0^T \frac{18\nu_f}{d^2(\rho_p/\rho_f)}\, dt$$

$$\ln\left(\frac{U_\infty - V}{U_\infty}\right) = -\frac{18\nu_f}{d^2(\rho_p/\rho_f)} T \Rightarrow V(t) = U_\infty\left(1 - e^{-\frac{t}{\tau_p}}\right)$$

Where $\tau_p = \dfrac{d^2(\rho_p/\rho_f)}{18\nu_f}$



If we introduce the particle with its gravitational terminal velocity:



$$m_p \frac{d\vec{V}_p}{dt} = (m_p - m_f)\vec{g} + \vec{D}rag$$

$$m_p \frac{dV_{px}}{dt} = D_x \; ; \quad m_p \frac{dV_{py}}{dt} = -(m_p - m_f)g + D_y$$

What are D_x and D_y? $\vec{D} = \frac{1}{2}\rho_f |\vec{U}_\infty - \vec{V}_p|(\vec{U}_\infty - \vec{V}_p) C_D \frac{\pi d^2}{4}$

$$\left.\begin{aligned} D_x &= \frac{1}{2}\rho_f |\vec{U}_\infty - \vec{V}_p|(U_\infty - V_{px}) C_D \frac{\pi d^2}{4} \\ D_y &= \frac{1}{2}\rho_f |\vec{U}_\infty - \vec{V}_p| V_t C_D \frac{\pi d^2}{4} \end{aligned}\right\}$$

Assuming that, initially $\vec{V}_p = 0\vec{i} - V_t\vec{j}$

At terminal velocity $\frac{dV_{py}}{dt} = 0 = -\frac{\pi d^3}{6}(\rho_p - \rho_f)g + \frac{1}{2}\rho_f |\vec{U}_\infty - \vec{V}_p| V_t C_D \frac{\pi d^2}{4}$

$|\vec{U}_\infty - \vec{V}_p| = \sqrt{(U_\infty - V_{px})^2 + V_t^2}$ and the maximum value of the Reynolds number happens at $t=0$ when $V_{px} = 0$

if $Re_p = \frac{d\sqrt{U_\infty^2 + V_t^2}}{\nu_f} \ll 1 \Rightarrow C_D = \frac{24}{Re_p}$ and we get

$$\frac{1}{2}\rho_f \cancel{|\vec{U}_\infty - \vec{V}_p|} \cdot V_t \cdot \frac{24}{\frac{\cancel{|\vec{U}_\infty - \vec{V}_p|} \cdot \cancel{d}}{\nu_f}} \cdot \frac{\cancel{\pi} \cancel{d}^2}{4} = \frac{\cancel{\pi} d^3}{6}(\rho_p - \rho_f) g$$

$$V_t = \frac{\left(\rho_p/\rho_f - 1\right) d^2 \cdot g}{18\,\nu_f} \underset{\frac{\rho_p}{\rho_f} \gg 1}{\approx} \tau_p \cdot g$$

and
$$\frac{\cancel{\pi} d^{\cancel{3}}}{6}\rho_p \frac{dV_{px}}{dt} = \frac{1}{2}\rho_f \cancel{|\vec{U}_\infty - \vec{V}_p|}\left(U_\infty - V_{px}\right)\frac{24}{\frac{\cancel{|\vec{U}_\infty - \vec{V}_p|} \cdot \cancel{d}}{\nu_f}} \cdot \frac{\cancel{\pi} d^2}{4}$$

$$\frac{dV_{px}}{\left(U_\infty - V_{px}\right)} = \frac{18\,\nu_f}{\left(\rho_p/\rho_f\right) d^2} \Rightarrow V_{px} = U_\infty \left[1 - e^{-\frac{t}{\frac{(\rho_p/\rho_f) d^2}{18\,\nu_f}}}\right]$$

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Oseen approximation

The inertial term will become more important as the distance from the sphere increases, so even if $R_e = \frac{\rho U_\infty R}{\mu} \ll 1$ at a distance r such that $R_e^* = \frac{\rho V_\infty r}{\mu} = 1$, that is at a distance $r = \frac{R}{R_e}$, the inertial term is of the same order as the viscous term.

The convective term can be rewritten as

$\rho(-\vec{V}_\infty \cdot \nabla\vec{v} + \vec{v}\cdot\nabla\vec{v})$ by decomposing the velocity into 2 components: $\vec{V}_\infty$ the free stream and $\vec{v}$ the perturbation due to the sphere. Obviously $\vec{v}\cdot\nabla\vec{v}$ is much smaller than $\vec{V}_\infty \cdot \nabla\vec{v}$ so we can take the term as the dominant part of the inertial term.

$$\left.\begin{aligned} &\nabla\cdot\vec{v} = 0 \\ &-\rho\vec{V}_\infty\cdot\nabla\vec{v} = -\nabla p + \mu\nabla^2\vec{v} \end{aligned}\right\}$$ Removed the non-linear $\vec{v}\cdot\nabla\vec{v}$

Approximate solution

$$\Psi = U_\infty R^2\left[-\frac{R}{4r}\sin^2\theta + 3(1-\cos\theta)\frac{1-e^{-\frac{1}{4}Re(1+\cos\theta)\frac{r}{R}}}{Re}\right]$$

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Near the sphere, where $\frac{r}{R} \cdot Re \ll 1$ we get

$$\Psi = U_\infty R \left[-\frac{R}{4r}\sin^2\theta + 3(1-\cos\theta)\frac{\cancel{1}-(\cancel{1}-x+x^2-}{Re} \right.$$

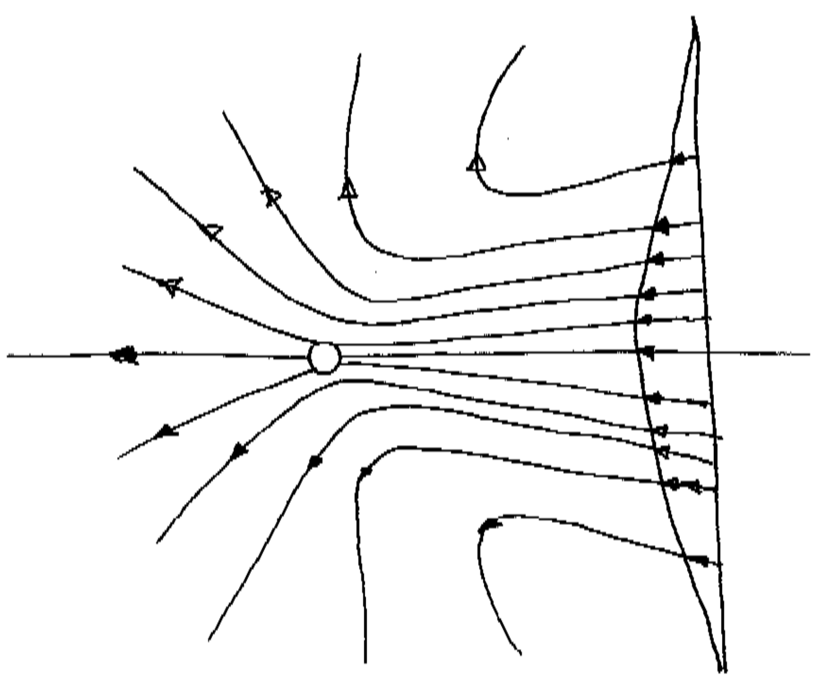
$$\Psi = U_\infty R^2 \left[-\frac{R}{4r}\sin^2\theta + \frac{3(1-\cos^2\theta)\, Re\, r/R}{4\, Re} - \frac{3(1-\cos\theta)\overset{(1+\cos\theta)}{}\, Re\, r/R}{16\, Re} \right.$$

$$\Psi = U_\infty R^2 \sin^2\theta \left\{ -\frac{R}{4r} + \frac{3}{4}\frac{r}{R} + O\left(Re\, r/R\right) \right\}$$

The resulting drag coefficient is:

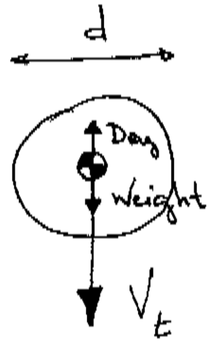
$$C_D = \frac{24}{Re}\left(1 + \frac{3}{16} Re\right)$$

Stokes Law — Oseen contribution



Streamlines due to the motion of a sphere relative to an stationary observer. Note: not pathlines

Terminal velocity of a particle in a stationary / laminar flow.



$$m_p \frac{d\vec{V}}{dt} = \begin{pmatrix} m_p \\ -m_f \end{pmatrix}\vec{g} - \vec{D}_{rag} = 0$$ at termin velocity.

$$\frac{\cancel{\pi}}{6} d^{\cancel{3}}\left(\rho_p - \rho_f\right) g = \frac{1}{2}\rho_f V_t^2 C_D \frac{\cancel{\pi} \cancel{d^2}}{4}$$

$$V_t^2 = \frac{4}{3}\frac{d\left(\rho_p-\rho_f\right)}{C_D \rho_f} g$$

What is C_D? For Stokes flow: $C_D = \frac{24}{Re}$

$$V_t^2 = \frac{4}{3}\frac{d\left(\rho_p-\rho_f\right)}{\frac{24}{Re}\cdot\rho_f} g \; ; \quad \text{but } Re = \frac{V_t \cdot d}{\nu} = \frac{\rho_f V_t d}{\mu_f}$$

$$V_t^{\cancel{2}} = \frac{\cancel{4}}{3}\frac{d\left(\rho_p-\rho_f\right)}{\frac{24}{\cancel{\rho_f}\cdot V_t \cdot d/\mu}\cdot\cancel{\rho_f}} g = \frac{1}{18}\frac{d^2\left(\rho_p-\rho_f\right)}{\mu}\cancel{V_t} g$$

$$V_t = \frac{1}{18}\frac{d^2\left(\rho_p-\rho_f\right)g}{\mu} = \frac{1}{18}\frac{d^2 g}{\nu_f}\left(\frac{\rho_p}{\rho_f}-1\right)$$

If $Re \approx 1$ however we can use Oseen $C_D = \frac{24}{Re}\left(1+\frac{3}{16}R\right.$

$$V_t^2 = \frac{4}{3}\frac{d\left(\rho_p/\rho_f - 1\right)g}{\frac{24}{V_t d/\nu_f}+\frac{9}{2}} \Rightarrow \frac{9}{2}V_t^2 + \frac{24}{d/\nu_f}V_t = \frac{4}{3}d\left(\rho_p/\rho_f-1\right)g$$

$$V_t = \frac{-\frac{24}{d/\nu_f} \pm \sqrt{\left(\frac{24}{d/\nu_f}\right)^2 + 24\, d\left(\rho_p/\rho_f-1\right)g}}{9}$$

The negative value is spurious:

$$V_t = \frac{24}{9d/\nu_f}\left(\sqrt{1+\frac{\left(\rho_p/\rho_f-1\right)g\, d^3/\nu_f^2}{24}} - 1\right)$$

For any expressoon of $C_D = f(Re)$ more complicated tha this, the problem of obtaining the terminal velocity becomes iterative.

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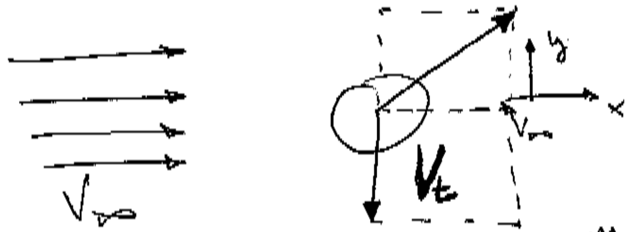

$$m_p \frac{d\vec{V}}{dt} = (m_p - m_f)\vec{g} + \vec{D}rag$$

$$m_p \frac{dV_x}{dt} = + D_x$$

$$m_p \frac{dV_y}{dt} = -(m_p - m_f)g + D_y$$

What is D_x and D_y? $\vec{D} = \frac{1}{2}\cdot\rho\cdot\underbrace{\left|\vec{V}_f - \vec{V}_p\right|\left(\vec{V}_f - \vec{V}_p\right)}\cdot C_D\cdot S'$

magnitude is ΔV^2
direction is $\vec{V}_f - \vec{V}_p$

if $\frac{dV_y}{dt} = 0 \rightarrow \frac{\pi}{6}d^3(\rho_p - \rho_f)g = \frac{1}{2}\rho_f\sqrt{V_\infty^2 + V_t^2}\cdot V_t\cdot C_D\cdot S'$

and $C_D = \frac{24}{\frac{|\vec{V}_f - \vec{V}_p|d}{\nu_f}} = \frac{24}{\frac{\sqrt{V_\infty^2 + V_t^2}\cdot d}{\nu_f}}$ so $\frac{\cancel{\pi}}{6}d^3(\rho_p - \rho_f)g = \frac{1}{2}\rho_f \cancel{\sqrt{V_\infty^2+V_t^2}}\cdot V_t \cdot \frac{24}{\frac{\sqrt{V_\infty^2 + V_t^2}}{\nu_f}}$

$$\boxed{V_t = \frac{1}{18}\left(\frac{\rho_p}{\rho_f} - 1\right)\frac{d^2 g}{\nu_f}}$$

Same as in a stagnant fluid EXCEPT IF

$$Re_p = \frac{|\vec{V}_f - \vec{V}_p|d}{\nu_f} = \frac{\sqrt{V_\infty^2 + V_t^2}\,d}{\nu_f} > 1$$

then $C_D \neq \frac{24}{Re}$ and the expression changes.