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Equation of motion for a small rigid sphere in a nonuniform flow

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The forces on a small rigid sphere in a nonuniform flow are considered from first prinicples in order to resolve the errors in Tchen's equation and the subsequent modified versions that have since appeared. Forces from the undisturbed flow and the disturbance flow created by the presence of the sphere are treated separately. Proper account is taken of the effect of spatial variations of the undisturbed flow on both forces. In particular the appropriate Faxen correction for unsteady Stokes flow is derived and included as part of the consistent approximation for the equation of motion.

I. INTRODUCTION

Since Tchen[1] first proposed an equation for the motion of a rigid sphere in a nonuniform flow, several papers have appeared correcting or modifying terms in the equation. Originally Basset[2] and later Boussinesq[3] and Oseen[4] examined the motion of a sphere settling out under gravity in a fluid that was otherwise at rest. The disturbance flow produced by the motion of the sphere was assumed to be at sufficiently low Reynolds number that the fluid force on the sphere could be calculated from the results of unsteady Stokes flow. Tchen extended this work first to a sphere settling under gravity in a fluid with an unsteady but uniform flow and then second to an unsteady and nonuniform flow, with a view to application to turbulent flows. Unfortunately Tchen's second extension was somewhat *ad hoc* and contained several errors.

Corrsin and Lumley[5] pointed out some of the inconsistencies in Tchen's equation, and, in particular, emphasized the role of the pressure gradient of the basic flow in contributing also to the net fluid force on the particle. For a small rigid sphere of radius a and mass m_p instantaneously centered at $\mathbf{Y}(t)$ and moving with velocity $\mathbf{V}(t)$, the equation proposed by Corrsin and Lumley is

$$\begin{aligned} m_p \frac{dV_i}{dt} &= m_F \left(\frac{Du_i}{Dt} - \nu \nabla^2 u_i \right) \Bigg|_{\mathbf{Y}(t)} \\ &\quad - \frac{1}{2} m_F \frac{d}{dt} \{ V_i(t) - u_i[\mathbf{Y}(t), t] \} \\ &\quad - 6\pi a\mu \Big(V_i(t) - u_i[\mathbf{Y}(t), t] \\ &\quad + a \int_{-\infty}^{t} d\tau \frac{d/d\tau\{ V_i(\tau) - u_i[\mathbf{Y}(\tau), \tau] \}}{[\pi\nu(t-\tau)]^{1/2}} \Big) \\ &\quad + (m_p - m_F) g_i. \end{aligned} \tag{1}$$

The undisturbed flow field is $u_i(\mathbf{x}, t)$, m_F is the mass of fluid displaced by the sphere, and dynamic viscosity and kinematic viscosity are μ and ν, respectively. In the equation above it is important to note the distinction between the two different time derivatives. The derivative d/dt is used here to denote a time derivative following the moving sphere, so that

$$\frac{d}{dt} u_i[\mathbf{Y}(t), t] = \left(\frac{\partial u_i}{\partial t} + V_j \frac{\partial u_i}{\partial x_j} \right)_{\mathbf{x} = \mathbf{Y}(t)} \tag{2}$$

is the time derivative of the fluid velocity. The derivative D/Dt is used by contrast to denote the time derivative following a fluid element, and

$$\frac{Du_i}{Dt}\Bigg|_{\mathbf{Y}(t)} = \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right)_{\mathbf{x} = \mathbf{Y}(t)} \tag{3}$$

is the fluid acceleration as observed at the instantaneous center of the sphere. The terms on the right-hand side of Eq. (1) then correspond in turn to the effects of pressure gradient of the undisturbed flow, added mass, viscous Stokes drag, augmented viscous drag from the Basset history term, and buoyancy. The pressure gradient term in (1) was written in this form on the assumption that the undisturbed flow is incompressible and satisfies

$$\frac{Du_i}{Dt} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + g_i + \nu \nabla^2 u_i. \tag{4}$$

The equation of Corrsin and Lumley is still not consistent in that the effects of pressure gradient of the undisturbed flow have been singled out over the effects of viscous shear stress when in fact the two may well be comparable. This point was noted by Buevich.[6] Buevich went back to the original Basset–Boussinesq–Oseen equation and used a change of reference frame to a coordinate system moving with the particle to derive a new equation of motion. He concluded that the first term on the right-hand side of (1) should be replaced by a term

$$m_F \frac{d}{dt} \{ u_i[\mathbf{Y}(t), t] \}. \tag{5}$$

Riley[7] on the other hand used a similar analysis and concluded that the term should be

$$m_F \frac{Du_i}{Dt}\Bigg|_{\mathbf{x} = \mathbf{Y}(t)}. \tag{6}$$

This latter result is physically more realistic. For a small sphere, small compared to the scale of the spatial variations of the undisturbed flow, the effect of the undisturbed fluid stresses both from pressure and viscosity is to produce the same net force as would act on a fluid sphere of the same size.

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This force must equal the product of fluid mass and local fluid acceleration as given by (6). The expression (5) has no such dynamical significance.

Other papers have appeared on the subject of Tchen's equation, e.g., Soo[8] and Gitterman and Steinberg,[9] giving different modifications. Our aim in this paper is to give a rational derivation for the equation of motion of a small sphere with relative motion of low Reynolds number, and to resolve the appropriate form it should take. In particular the forces due to nonuniform flow will be re-examined. It is well known[10] that velocity curvature will modify the drag force on a sphere at low Reynolds number. For steady Stokes flow this correction is given by the Faxen[11] relations so that the force on the sphere produced by the disturbance flow around the sphere is

$$\mathbf{F} = 6\pi a\mu\{\mathbf{u}[\mathbf{Y}(t), t] - \mathbf{V}(t)\} + \mu\pi a^3(\nabla^2 u_i)_{\mathbf{Y}(t)}, \quad (7)$$

to first approximation. If pressure gradient forces and similar effects are to be included then the modified Stokes flow should also be considered since its contribution is of the same magnitude. In the Stokes flow regime velocity shear does not produce a resultant force, and there is no "lift" or side force.[12]

In Sec. II the problem for the disturbance flow around a rigid Stokes sphere in a nonuniform undisturbed flow is formulated and the equation of motion for the sphere set up following the approach of Riley.[7] In Sec. III the force produced by the disturbance flow is calculated for unsteady Stokes flow generalizing the results of Basset[2] and extending the work of Burgers.[13] Finally in Sec. IV the significance of the various effects included is discussed and compared with some others that are neglected. It is important to appreciate the limitations on the validity of Tchen's equation especially in turbulent flows, since errors may accumulate with time before a statistically asymptotic state is reached.

II. EQUATIONS OF MOTION

The problem considered then is that of a small rigid sphere of radius a located at $\mathbf{Y}(t)$ in a fluid flow which in the absence of the particle is $\mathbf{u}(\mathbf{x}, t)$. The presence of the particle and its motion through the fluid will modify the flow locally and lead to a new flow field $\mathbf{v}(\mathbf{x}, t)$. This modified flow must satisfy the conditions:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \rho g - \nabla p + \mu\nabla^2\mathbf{v}, \quad (8a)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (8b)$$

$$\mathbf{v} = \mathbf{V} + \mathbf{\Omega}\times[\mathbf{x} - \mathbf{Y}(t)] \text{ on the sphere,} \quad (8c)$$

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) \text{ as } |\mathbf{x} - \mathbf{Y}(t)| \to \infty, \quad (8d)$$

for an incompressible flow of uniform density. The no-slip condition (8c) is applied on the sphere so that locally the fluid velocity matches the particle velocity $\mathbf{V}(t)$ and angular velocity $\mathbf{\Omega}(t)$. If σ_{ij} is the fluid stress tensor

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad (9)$$

then the equation of motion for the spherical particle is

$$m_p \frac{dV_i}{dt} = m_p g_i + \oint_s \sigma_{ij} n_j dS, \quad (10)$$

where the surface integral is over the surface of the sphere and $\mathbf{n}$ is the unit outward normal. The problem is to evaluate the fluid stress tensor on the sphere.

To solve the fluid mechanical problem it is convenient to made a change of coordinates to a frame moving with the center of the sphere. The changes of variable are

$$\mathbf{z} = \mathbf{x} - \mathbf{Y}(t), \quad t = t, \quad (11a)$$

$$\mathbf{w}(\mathbf{z}, t) = \mathbf{v}(\mathbf{x}, t) - \mathbf{V}(t). \quad (11b)$$

The conditions (8) now become

$$\rho\left(\frac{\partial w_i}{\partial t} + w_j\frac{\partial w_i}{\partial z_j}\right) = \rho\left(g_i - \frac{dV_i}{dt}\right) - \frac{\partial p}{\partial z_i} + \mu\frac{\partial^2 w_i}{\partial z_j \partial z_j}, \quad (12a)$$

$$\frac{\partial w_i}{\partial z_i} = 0, \quad (12b)$$

$$\mathbf{w} = \mathbf{\Omega}\times\mathbf{z} \quad \text{for } |\mathbf{z}| = a, \quad (12c)$$

$$\mathbf{w} = \mathbf{u} - \mathbf{V} \quad \text{as } |\mathbf{z}| \to \infty. \quad (12d)$$

The stress tensor can be expressed similarly as

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial w_i}{\partial z_j} + \frac{\partial w_j}{\partial z_i}\right). \quad (13)$$

In order to take advantage of small parameters in this problem it is further convenient to separate the flow field into two parts $\mathbf{w}^{(0)}$ and $\mathbf{w}^{(1)}$; $\mathbf{w}^{(0)}$ being the undisturbed flow without application of the boundary condition (12c),

$$\mathbf{w}^{(0)} = \mathbf{w} - \mathbf{w}^{(1)} = \mathbf{u} - \mathbf{V},$$

and $\mathbf{w}^{(1)}$ the disturbance flow set up by the particle. The two component flows satisfy the equations of motion

$$\rho\left(\frac{\partial w_i^{(0)}}{\partial t} + w_j^{(0)}\frac{\partial w_i^{(0)}}{\partial z_j}\right) = \rho\left(g_i - \frac{dV_i}{dt}\right) - \frac{\partial p^{(0)}}{\partial z_i} + \mu\frac{\partial^2 w_i^{(0)}}{\partial z_j \partial z_j}, \quad (14)$$

$$\rho\left(\frac{\partial w_i^{(1)}}{\partial t} + w_j^{(0)}\frac{\partial w_i^{(1)}}{\partial z_j} + w_j^{(1)}\frac{\partial w_i^{(0)}}{\partial z_j} + w_j^{(1)}\frac{\partial w_i^{(1)}}{\partial z_j}\right) = -\frac{\partial p^{(1)}}{\partial z_i} + \mu\frac{\partial^2 w_i^{(1)}}{\partial z_j \partial z_j}, \quad (15)$$

and each separately satisfies incompressibility (12b). If W_0 is taken as a representative velocity scale for $(\mathbf{V} - \mathbf{u})$, the velocity of the sphere relative to that of the surrounding fluid, and a Reynolds number is defined by aW_0/ν, then a scale analysis of (15) shows that for low Reynolds number the advective terms may be neglected leading to a problem in unsteady Stokes flow. The equation for the disturbance flow reduces to

$$\rho\frac{\partial w_i^{(1)}}{\partial t} = -\frac{\partial p^{(1)}}{\partial z_i} + \mu\frac{\partial^2 w_i^{(1)}}{\partial z_j \partial z_j}, \quad (16)$$

provided that

$$aW_0/\nu \ll 1 \text{ and } (a^2/\nu)(U_0/L) \ll 1, \quad (17)$$

where L is a representative differential length scale of the undisturbed flow, and U_0/L is the scale for the corresponding velocity gradient. The boundary conditions on the disturbance flow $\mathbf{w}^{(1)}$ are

$$\mathbf{w}^{(1)} = -(\mathbf{u}-\mathbf{V}) + \boldsymbol{\Omega}\times\mathbf{z} \quad \text{on } |\mathbf{z}| = a, \tag{18}$$

$$|\mathbf{w}^{(1)}| \to 0 \quad \text{as } |\mathbf{z}| \to \infty. \tag{19}$$

The estimation of the fluid force on the sphere now rests on finding the surface stress produced by the unsteady Stokes flow governed by (16), (18), and (19). An estimate is only sought to a first approximation in the limit of zero Reynolds number. No attempt is made to evaluate Oseen corrections which will be an order of magnitude smaller, and so only the inner Stokes flow problem will be treated in the following section. The time derivative is retained in (16) as there are several examples, e.g., a particle falling from rest under gravity, where the unsteady term is important.

Besides the disturbance flow there is also a contribution to the fluid force from the undisturbed flow $\mathbf{w}^{(0)}$, which may be found quite generally and without specific assumptions about low Reynolds number such as condition (17) used to derive (16). From (13) and (10) the contribution to the fluid force on the particle from $\mathbf{w}^{(0)}$ is $\mathbf{F}^{(0)}$

$$F_i^{(0)} = \oint_s n_j\left[-p^{(0)}\delta_{ij} + \mu\left(\frac{\partial w_i^{(0)}}{\partial z_j} + \frac{\partial w_j^{(0)}}{\partial z_i}\right)\right]. \tag{20}$$

This may be converted to a volume integral and approximately evaluated as

$$F_i^{(0)} = \frac{4}{3}\pi a^3\left(-\frac{\partial p^{(0)}}{\partial z_i} + \mu\frac{\partial^2 w_i^{(0)}}{\partial z_j \partial z_j}\right), \tag{21}$$

on the assumption that the terms in parentheses are nearly uniform over the sphere, provided the sphere is sufficiently small. This implies that the pressure gradient is approximately uniform over the sphere and that the velocity $w_i^{(0)}$ can be expressed as

$$w_i^{(0)}(\mathbf{z},t) = w_i^{(0)}(0,t) + z_j\frac{\partial w_i^{(0)}}{\partial z_j} + \frac{1}{2}z_j z_k\frac{\partial^2 w_i^{(0)}}{\partial z_j \partial z_k} + O(a^3/L^3), \tag{22}$$

in the neighborhood of the sphere. This assumption is valid if the size of the sphere is small composed to the length scale of variations in the undisturbed flow so that

$$a/L \ll 1. \tag{23}$$

In this manner linear variations in $\sigma_{ij}^{(0)}$ are included. The result (21) may be simplified by use of the momentum equation (14) for the undisturbed flow

$$F_i^{(0)} = -m_F g_i + m_F\left(\frac{dV_i}{dt} + \frac{\partial w_i^{(0)}}{\partial t} + w_j^{(0)}\frac{\partial w_i^{(0)}}{\partial z_j}\right), \tag{24}$$

or in terms of the undisturbed flow $\mathbf{u}(\mathbf{x}, t)$ in the original frame of reference

$$F_i^{(0)} = -m_F g_i + m_F\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right)\bigg|_{\mathbf{Y}(t)}. \tag{25}$$

This is the result first given by Riley.[7]

In summary then the equation for particle motion (10) is

$$m_p\frac{dV_i}{dt} = (m_p - m_F)g_i + m_F\frac{Du_i}{Dt}\bigg|_{\mathbf{Y}(t)} + F_i^{(1)}, \tag{26}$$

assuming the condition (23). The fluid force $\mathbf{F}^{(1)}$ from the disturbance flow $\mathbf{w}^{(1)}$ around the sphere is determined here for the particular limit of low Reynolds number for the disturbance flow (17) and found by solving Eqs. (16), (18), and (19) to evaluate

$$F_i^{(1)} = \oint_s\left[-p^{(1)}\delta_{ij} + \mu\left(\frac{\partial w_i^{(1)}}{\partial z_j} + \frac{\partial w_j^{(1)}}{\partial z_i}\right)\right]n_j\,dS. \tag{27}$$

The important point to note though is that the consistent inclusion of pressure gradient terms and forces from the undisturbed flow requires the approximate expansion (22) for $\mathbf{w}^{(0)}$ close to the sphere. To the same degree of approximation this representation must be used in evaluating the boundary conditions (18) on the surface of the sphere and this will alter the usual Basset–Boussinesq–Oseen results.

III. FORCE FROM THE UNSTEADY STOKES FLOW

A detailed solution of the disturbance flow $\mathbf{w}^{(1)}$ is not necessary to evaluate the force on the sphere. Following Pérès,[14] who gave a derivation for the Faxen relation for steady Stokes flow, a symmetry relationship for unsteady Stokes flow will be obtained and used in conjunction with the fundamental point force flow of Burgers[13] to evaluate the disturbance flow force $\mathbf{F}^{(1)}$. The analysis will be in terms of Laplace transforms with respect to time and the notation

$$\hat{f}(s) = \int_0^\infty e^{-st}f(t)\,dt, \tag{28}$$

will be used.

Consider two unsteady Stokes flow fields $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{u}'(\mathbf{x}, t)$ in some volume V bounded by a surface S, and each satisfying

$$\rho\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x_j}\sigma_{ij} = 0, \tag{29a}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{29b}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{29c}$$

Consideration then of the volume integral of the Laplace transforms

$$\int_V\left(\hat{u}_i\frac{\partial}{\partial x_j}\hat{\sigma}'_{ij} - \hat{u}'_i\frac{\partial}{\partial x_j}\hat{\sigma}_{ij}\right)dV, \tag{30}$$

leads to the following result

$$\oint_s dS\,(\hat{u}_i\hat{\sigma}'_{ij} - \hat{u}'_i\hat{\sigma}_{ij})m_j = \int_V dV\rho\left[u_i(\mathbf{x},0)\hat{u}'_i(\mathbf{x},s) - u'_i(\mathbf{x},0)\hat{u}_i(\mathbf{x},s)\right], \tag{31}$$

where $\mathbf{m}$ is the outward unit normal. This symmetry relationship can be applied to the disturbance flow around the sphere. Provided both $\mathbf{u}$ and $\mathbf{u}'$ tend to zero far from the sphere the integral over the surface at infinity will make no contribution and the surface integral reduces to an integral over the surface of the sphere. The flow field $\mathbf{u}'$ is required to satisfy the boundary conditions

$$|\mathbf{u}'| \to 0 \quad \text{as } |x| \to \infty, \tag{32a}$$

$$\hat{\mathbf{u}}'(\mathbf{x},s) = (1,0,0) \quad \text{on } |\mathbf{x}| = a, \tag{32b}$$

$$\mathbf{u}'(\mathbf{x}, t \leqslant 0) = 0, \tag{32c}$$

the origin being at the center of the sphere. The flow $\mathbf{u}'$ corresponds to the fluid motion resulting from the sphere starting from rest and having an impulsive velocity $[\delta(t),0,0]$. The symmetry relationship (31) then gives a method for evaluating the force on the sphere from the $\mathbf{u}$ flow field

$$\hat{F}_1 = \oint_s \hat{\sigma}_{ij} n_j \, dS = \oint_s \hat{u}_i \hat{\sigma}'_{ij} n_j \, dS + \int_V \rho u_i(\mathbf{x},0)\hat{u}'_i(\mathbf{x},s) dV, \tag{33}$$

where $\mathbf{n}$ is now the unit normal out of the sphere. Similar results hold for the other components.

The flow $\mathbf{u}'$ can be found by adapting the unsteady Stokes flow solution of Burgers.[13] If $\mathbf{e}^{(1)}$ denotes the unit vector (1,0,0)

$$\hat{\mathbf{u}}' = (\mathbf{e}^{(1)} \cdot \nabla)\nabla\hat{\psi} - \mathbf{e}^{(1)}\nabla^2\hat{\psi}, \tag{34}$$

$$\hat{\psi}(r,s) = Q_1/\rho s r + (Q_2/r)\exp(-\lambda r), \tag{35}$$

where $\lambda^2 = \rho s/\mu$, $\mathrm{Re}(\lambda)$ is taken to be positive, and r is the radial disturbance from the origin. The boundary conditions give

$$Q_1 = \tfrac{3}{2} a\mu(1 + \lambda a) + \tfrac{1}{2}a^3 \rho s, \tag{36}$$

$$Q_2 = -\tfrac{3}{2}(a\mu/\rho s)\exp(\lambda a), \tag{37}$$

and on the surface of the sphere $(r = a)$

$$\hat{\sigma}'_{ij} n_j = -\tfrac{1}{2} a\rho s x_i x_j e_j^{(1)}/a^2 - \tfrac{3}{2}\mu/a(1 + \lambda a)e_i^{(1)}. \tag{38}$$

For a flow field $\mathbf{u}$ that is initially zero everywhere, and on the sphere has the form

$$\hat{u}_i(\mathbf{x},s) = \hat{A}_i + x_j \hat{B}_{ij} + \tfrac{1}{2} x_j x_k \hat{C}_{ijk}, \tag{39}$$

$$\hat{F}_1 = -\hat{A}_1[6\pi a\mu(1 + \lambda a) + \tfrac{1}{2} m_F s] - a^2[(\hat{C}_{1jj} + 2\hat{C}_{jj1}) \times \pi\mu a(1 + \lambda a) + \tfrac{1}{20} m_F s\hat{C}_{1jj}]. \tag{40}$$

This Laplace transform can be readily inverted to give the force as a function of time. The term in (λa) leads to the usual Basset history integral.

IV. EQUATION OF MOTION FOR THE SPHERE

The resultant force due to the disturbance flow derived above may now be included in the equation of particle motion (26). Comparison of the boundary conditions (18) and (39) shows that

$$A_i = V_i(t) - u_i[\mathbf{Y}(t), t], \tag{41}$$

$$C_{ijk} = -\left.\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|_{\mathbf{Y}(t)}, \tag{42}$$

so that the final form of the equation of particle motion after inversion of the transforms is

$$m_p \frac{dV_i}{dt} = (m_p - m_F)g_i + m_F \left.\frac{Du_i}{Dt}\right|_{\mathbf{Y}(t)} - \frac{1}{2} m_F \frac{d}{dt}\{V_i(t) - u_i[\mathbf{Y}(t), t] - \tfrac{1}{10} a^2\nabla^2 u_i|_{\mathbf{Y}(t)}\} - 6\pi a\mu\{V_i(t) - u_i[\mathbf{Y}(t), t] - \tfrac{1}{6} a^2\nabla^2 u_i|_{\mathbf{Y}(t)}\} - 6\pi a^2\mu \int_0^t d\tau \left(\frac{d/d\tau\{V_i(\tau) - u_i[\mathbf{Y}(\tau),\tau] - \tfrac{1}{6}a^2\nabla^2 u_i|_{\mathbf{Y}(\tau)}\}}{[\pi\nu(t-\tau)]^{1/2}}\right). \tag{43}$$

The initial conditions are that the sphere is introduced at $t = 0$ and that there is no disturbance flow $\mathbf{w}^{(1)}$ prior to this. The restrictions on scales are given by (17) and (23).

The derivation of (43) is based on a consistent treatment of the inertia and pressure gradient terms for a sphere in a nonuniform flow field. The inclusion of velocity gradients leads to modifications of the added mass terms, the Stokes drag, and the Basset history term due to curvature in the velocity profile; while in the low Reynolds number limit there is no force due to shear or particle spin. Besides the Faxen terms, the equation of motion differs from previous versions in the form of the fluid acceleration term $m_F\, Du_i/Dt$ as opposed to $m_F\, du_i/dt$ given by Buevich.[6] In general the values of these two derivatives, one following a fluid element and the other following the particle can differ substantially. However in the context of the low Reynolds number approximation made here the two derivatives are approximately the same. Specifically the difference between the two terms is equal to

$$m_F\, w_j^{(0)} \frac{\partial w_i^{(0)}}{\partial z_j}, \tag{44}$$

which compared to the dominant Stokes drag term is $O(a^2 U_0/L\nu)$ and small by condition (17). This difference is also the same order of magnitude as one of the terms neglected in deriving Eq. (16) from (15).

It should also be remarked that a similar question arises over the form of the added mass term in (43). For potential flow about a spherical bubble when the ambient flow is irrotational and nonuniform Auton[15,16] has shown that the usual added mass form

$$\frac{1}{2} m_F \frac{d}{dt}\{V_i(t) - u_i[\mathbf{Y}(t), t]\}, \tag{45}$$

is incorrect and should be

$$\frac{1}{2} m_F \left(\frac{dV_i}{dt} - \left.\frac{Du_i}{Dt}\right|_{\mathbf{Y}(t)}\right). \tag{46}$$

Again there is no distinction here between these two results because of the low Reynolds number approximation used in evaluating the disturbance flow field force $\mathbf{F}^{(1)}$.

The restrictions (17) and (23) do constrain the importance of many of the terms retained in Eq. (43). In general the Stokes drag term is the most important and (43) often reduces to the statement that the particle velocity is approximately equal to the local fluid velocity, or, if the buoyancy forces are significant, to a balance of Stokes drag force with gravitational forces. There are specific examples though where other terms in (43) are important. For an aerosol the particle density is much greater than the fluid density so that, even if local fluid inertia is a negligible effect, the particle inertia term can be significant. The Basset history term and added mass term result from retaining the time derivative in the equation (16) for the disturbance flow. These are both important in the motion of a particle falling under gravity through still fluid from an initial state of rest as shown by Boggio[17] and Hjemfelt and Mockros,[18] and for a particle suspended in a fluid oscillating uniformly at high frequency. In the latter example the fluid oscillations limit the diffusion of vorticity away from the surface of the sphere, confining it to a thin Stokes layer. The Basset term then gives an augmented drag force. However the Reynolds number condition necessarily requires that the displacement amplitude of the relative motion of the particle in the fluid be small compared to the particle radius. For a particle falling from rest there is initially little relative motion and only a weak Stokes drag. The effects of gravity are balanced by inertia and history terms. Eventually the vorticity generated by the relative motion diffuses away from the sphere, but not until the vorticity has diffused several radii does the Stokes drag reach an equilibrium value. These points are reviewed by Clift *et al.*[19]

Significant time dependent effects can also be introduced through inhomogeneities of the underlying flow field, but again the conditions of low Reynolds number and small particle radius limit their effect. Some estimate of the magnitude of various terms in (43) can be made using the velocity and length scales given previously. The equation of particle motion is best viewed as providing the relative velocity of the particle in the fluid, and it is useful to define

$$\mathbf{W}(t) = \mathbf{V}(t) - \mathbf{u}[\mathbf{Y}(t), t]. \tag{47}$$

In terms of the relative velocity $\mathbf{W}$ the equation of particle motion is then

$$\left(m_p + \frac{1}{2} m_F\right)\frac{dW_i}{dt} + 6\pi a^2\mu \int_0^t d\tau \frac{dW_i}{d\tau}[\pi\nu(t-\tau)]^{-1/2} + 6\pi a\mu W_i$$
$$= -m_p \frac{du_i}{dt} + m_F \frac{Du_i}{Dt} + (m_p - m_F) g_i + a^3\pi\mu\nabla^2 u_i + \frac{1}{20} a^2 m_F \frac{d}{dt}(\nabla^2 u_i|_{\mathbf{Y}(t)}) + \pi\mu a^4 \int_0^t d\tau \frac{d}{d\tau}(\nabla^2 u_i|_{\mathbf{Y}(t)})$$
$$\times [\pi\nu(t-\tau)]^{-1/2}. \tag{48}$$

The terms on the right of Eq. (48) may be regarded as source terms for the relative velocity. Approximate results for $\mathbf{W}(t)$ may be derived depending on the accuracy required by the physical problem under consideration. In some instances an estimate of particle velocity is only required to within some percentage of the fluid velocity scale U_0, in which case many terms are negligible. In other instances the longer term drift of the particle is required and the relative velocity has to be estimated to within a certain fraction of itself, or if the particle is settling under gravity, to within a fraction of the mean settling velocity. In this case many of the terms in (48), such as fluid acceleration, are important. The accuracy to which $\mathbf{W}(t)$ is estimated will also determine the time scale over which the equation of motion can be applied, as errors accumulate in time and in an inhomogeneous flow the particle can eventually drift into regions of very different flow characteristics.

Many effects are neglected in deriving the equation of motion. The sphere is assumed to be isolated and far from any boundary so that particle–particle interactions and particle–boundary interactions are excluded. This requires that the distance from the nearest particle or boundary is very much greater than the sphere radius. Effects of nonzero Reynolds number for the relative motion are also neglected. For steady motion these may be categorized as the Oseen correction to Stokes drag, the modified drag due to particle rotation, and the Saffman effect or side force due to the shear of the undisturbed flow. These effects though are all small compared to the basic Stokes drag term $[6\pi\mu a\mathbf{W}(t)]$ with the assumptions of low Reynolds number. Specifically the Oseen correction for rectilinear motion is $O(aW_0/\nu)$. Rubinow and Keller[20] have calculated the side force on a sphere rotating with angular velocity $\mathbf{\Omega}$ while translating with velocity $\mathbf{W}$ and shown the side force to be $O(a^2\Omega/\nu)$, this being due to the modified outer flow around the sphere. In shear flow the particle will tend to rotate with the surrounding fluid and the angular velocity should be comparable to the local velocity gradient U_0/L. So this correction is $O(a^2U_0/L\nu)$. The most critical effect is the side force due to shear. Saffman[21] showed that a sphere subjected to a uniform shear and rotating with the fluid while translating with velocity $\mathbf{W}$ experiences a side force of relative magnitude $O(a^2U_0/L\nu)^{1/2}$. This result depends on the shear being uniform not only in the immediate vicinity of the sphere but also throughout the outer region.

These estimates lead us to expect that the relative velocity $\mathbf{W}(t)$ can be evaluated from the particle motion equation to an accuracy of

$$\mathbf{W}(t) \sim W_0[1 + O(aW_0/\nu) + O(a^2U_0/L\nu)^{1/2}] + O[U_0(a^4/L^4)]. \tag{49}$$

The latter term comes from the next higher approximation for the local fluid velocity on the surface of the sphere. These conditions now allow us to gauge the relative importance of velocity gradients in the undisturbed flow on the relative motion of the particle. Referring back to (48) the fluid acceleration terms

$$\frac{du_i}{dt} \sim O(U_0^2/L) \sim \frac{Du_i}{Dt}, \tag{50}$$

so that the scale of relative motion W_0 due to particle inertia is

$$O\,[(m_p/m_F)(a^2U_0/\upsilon L)U_0],$$

while that due to fluid inertia is

$$O\,[(a^2U_0/\upsilon L)U_0].$$

Unless the particle is much denser than the surrounding fluid both of these estimates for W_0 are a small fraction of the local fluid velocity because of the shear Reynolds number condition. However for details of the relative motion they may be important, especially for long term considerations. The Faxen term gives a relative motion scale of $O\,(a^2U_0/L^2)$. Again this may be a small fraction of the local fluid velocity, but also it can be important to the details of relative motion.

In gauging these effects the inertia and history terms on the left of (48) were ignored. Both are significant at the initial instance that the particle starts to move relative to the fluid, much as for a particle falling through still fluid. But once the motion is established the acceleration of relative motion as the particle moves through the velocity gradient is $O\,[(U_0/L)W_0]$ and these terms are generally small. The added mass term is $O\,(a^2U_0/L\upsilon)$ compared to the Stokes drag while the history term is $O\,(a^2U_0/L\upsilon)^{1/2}$. Both are small from the shear Reynolds number condition, with the history term slightly more significant. The particle inertia term is again important if the particle is much denser than the surrounding fluid. Throughout this analysis it is apparent that the shear Reynolds number condition of (17) restricting the fluid velocity gradient eliminates many of the possible effects.

The equation of particle motion has also been applied to turbulent flows, as by Hjelmfelt and Mockros,[22] to estimate how well particle tracers follow the local fluid motion. However for a turbulent flow there is no single set of scales but rather a continuous spectrum of velocity and length scales which must be considered in any application of the equation of particle motion. The larger scale, more energetic, motions may be characterized by a root mean square (rms) velocity fluctuation scale u' and an integral length scale L; while the smaller scale, dissipative motions may be characterized in terms of the Kolmogorov microscales υ_k and η_k.[23] The ratios of these scales are

$$\upsilon_k/u' = O(R_\lambda^{-1/2}), \quad \eta_k/L = O(R_\lambda^{-3/2}), \tag{51}$$

based on the Reynolds number R_λ from the Taylor microscale λ, $u'\lambda/\upsilon$. The velocity gradients are dominated by the small scale motions and υ_k/η_k or equivalently u'/λ may be taken as a representative scale. The condition (23) on the size of the particle then necessitates

$$a/\eta_k \ll 1, \tag{52}$$

while the Reynolds number conditions (17) require, respectively, that

$$aW_0/\upsilon = O\,[(a/\eta_k)(W_0/\upsilon_k)] \ll 1, \tag{53}$$

$$a^2\upsilon_k/\eta_k\upsilon = O\,(a^2/\eta_k^2) \ll 1, \tag{54}$$

the possible side force due to the Saffman effect is $O\,(a/\eta_k)$ compared to the Stokes drag force.

These restrictions again limit the importance of several terms in (43) especially the added mass and history terms. For turbulent diffusion the advection of the particle by the larger scale eddies is the most important feature. It is only necessary to calculate the particle velocity to within some small fraction of u' and for sufficient time for the statistically asymptotic long term behavior to be reached. The equation (43) then reduces to a balance of Stokes drag, buoyancy forces, and possibly particle inertia. For more detailed studies, such as the velocity spectrum of the particle, other terms may be significant, especially for the small scale motions at higher frequency. Hjelmfelt and Mockros[22] in their analysis of velocity spectrum took a very high frequency as the upper limit of the spectrum based on measurements of the Eulerian frequency spectrum. While the particles may not move perfectly with the fluid it is more appropriate to use the Lagrangian frequency spectrum cutoff of υ_k/η_k if the scale of relative motion W_0 is smaller than υ_k, or to use W_0/η_k if the relative motion is larger. The Lagrangian frequency spectrum cutoff is much lower than the corresponding Eulerian cutoff as has been shown by Tennekes.[24] The Reynolds number conditions (53) and (54) ensure that the Stokes numbers based on these cutoff frequencies are large. For a cutoff of υ_k/η_k the Stokes number N_s is

$$N_s = (\nu\eta_k/\upsilon_k a^2)^{1/2} = (\eta_k/a),$$

or for a cutoff of W_0/η_k

$$N_s = (\nu\eta_k/W_0a^2)^{1/2} = [(\eta_k^2/a^2)(\upsilon_k/W_0)]^{1/2}.$$

The equation of particle motion (43) further can be modified to flows in which the undisturbed flow field is compressible, provided that we can still assume that the disturbance flow around the sphere is locally incompressible and of uniform density so that (16) still applies. The extra terms from compressible undisturbed flow are given by $\hat{C}_{jj\,1}$, etc., in Eq. (40) and can then be included in (43).

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