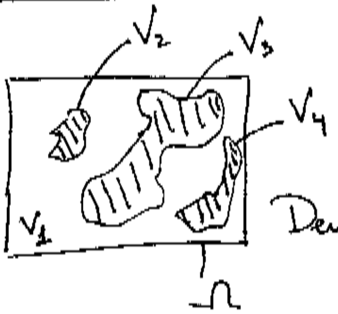
TWO FLUID EQUATIONS



Phases $i = 1, 4$

Density: $\hat{\rho}_i = \dfrac{1}{\Omega} \displaystyle\int_{V_i} \rho_i \, dV \qquad \rho = \sum_{i=1}^{N} \hat{\rho}_i$

Velocity: $\hat{\vec{v}}_i = \dfrac{1}{\hat{\rho}_i \Omega} \displaystyle\int_{V_i} \rho_i \vec{v}_i \, dV \qquad \vec{v} = \dfrac{1}{\rho} \sum_{i=1}^{N} \hat{\rho}_i \hat{\vec{v}}_i$

Internal Energy: $\hat{u}_i = \dfrac{1}{\hat{\rho} \Omega} \displaystyle\int_{V_i} \rho_i u_i \, dV; \quad u = \dfrac{1}{\rho} \sum_{i=1}^{N} \hat{\rho}_i$

We can define the material derivative for each phase:

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + \hat{\vec{v}}_i \cdot \nabla$$

and based on this, the conservation of mass for each phase will be:

$$\int_{\mathcal{V}} \frac{\partial \hat{\rho}_i}{\partial t} + \int_{\mathcal{S}} \rho_i \hat{\vec{v}}_i \cdot \vec{n} \, d\mathcal{S} = \int_{\mathcal{V}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} J_{ji} \right) d\mathcal{V}$$

$\mathcal{V}$ and $\mathcal{S}$ are the total control volume and the exterio surface (with normal vector pointing outwards $\vec{n}$) and J_{ji} is the flux of mass between phase j and phase i. $J_{ji} = -J$

Using the divergence theorem we get the continuity equation

$$\boxed{\frac{\partial \hat{\rho}_i}{\partial t} + \nabla \cdot \left(\hat{\rho}_i \hat{\vec{v}}_i \right) - \sum_{\substack{j=1 \\ j \neq i}}^{N} J_{ji} = 0}$$

Individual phase continuity equation.

$$\sum_{i=1}^{N} \rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

Continuity eq.

$$\frac{\partial}{\partial t}\left(\hat{\rho}_i \vec{\hat{v}}_i\right) + \nabla\cdot\left(\hat{\rho}_i \vec{\hat{v}}_i \vec{\hat{v}}_i\right) = \nabla\cdot\overline{\overline{\tau}}_i + \hat{\rho}_i \vec{g}_i + \sum_{j=1}^{N} \vec{P}_{ij}$$

$\overline{\overline{\tau}}_i$ is the stress tensor in the phase i

$\vec{P}_{ij}$ is the flux of momentum from phase j to i. It is defined so that $\vec{P}_{ii} = 0$ and $\vec{P}_{ij} = -\vec{P}_{ji}$

We define a relative velocity as:

$\vec{W}_i = \vec{\hat{v}}_i - \vec{v}$. $\vec{v} = \frac{1}{\rho}\sum \hat{\rho}_i \vec{\hat{v}}_i$ and, by definition,

$$\sum_{i=1}^{N} \hat{\rho}_i \vec{W}_i = 0$$

$$\sum_{i=1}^{N} \hat{\rho}_i \vec{\hat{v}}_i \vec{\hat{v}}_i = \sum_{i=1}^{N} \hat{\rho}_i \vec{v}\vec{v} + 2\underbrace{\left(\sum_{i=1}^{N} \hat{\rho}_i \vec{W}_i\right)}_{0}\vec{v} + \sum \hat{\rho}_i \vec{W}_i \vec{W}_i$$

$$\sum_{i=1}^{N} \hat{\rho}_i \vec{g}_i = \rho \vec{g} \quad ; \quad \sum_{i=1}^{N} \overline{\overline{\tau}}_i = \overline{\overline{\tau}}$$

The equation of conservation of momentum for all the phases

$$\frac{\partial}{\partial t}\left(\rho \vec{v}\right) + \nabla\cdot\left(\rho \vec{v}\vec{v}\right) = \nabla\cdot\left[\overline{\overline{\tau}} - \sum\left(\hat{\rho}_i \vec{W}_i \vec{W}_i\right)\right] + \rho\vec{g}$$

$$\hat{\rho}_i \frac{D_i \vec{\hat{v}}_i}{Dt} = \nabla\cdot\overline{\overline{\tau}}_i + \hat{\rho}_i \vec{g}_i + \sum_{j=1}^{N}\left(\vec{P}_{ij} - J_{ji}\vec{\hat{v}}_i\right)$$

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Energy Equation

$$\frac{\partial(\hat{\rho}_i \hat{E}_i)}{\partial t} + \nabla\cdot\left(\hat{\rho}_i \hat{E}_i \vec{\hat{v}}_i\right) = \nabla\cdot\left(\vec{\hat{W}}_i - \vec{\hat{q}}_i\right) + \hat{\rho}_i \hat{g}_i \cdot \vec{\hat{v}}_i + \sum_{\substack{j=1\\ j\neq i}}^{N} E_{ji}$$

$\hat{E}_i$ is the total energy for phase i, $\hat{E}_i = \hat{u}_i + \frac{|\vec{\hat{v}}_i|^2}{2}$

$\vec{\hat{W}}_i$ is the term that accounts for work done by moving external surfaces: $(-p\bar{\bar{I}} + \bar{\bar{\tau}}') \cdot \vec{\hat{v}}_i$

$\vec{\hat{q}}_i$ is the heat flux from phase i through the external surface.

E_{ji} is the exchange of energy between phase j and i

Adding all phases up, we get:

$$\rho E = \sum_{i=1}^{N} \hat{\rho}_i \hat{E}_i = \rho u + \frac{1}{2}\rho|\vec{v}|^2 + \sum_{i=1}^{N} \frac{\hat{\rho}_i |\vec{\hat{W}}_i|^2}{2}$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla\cdot\left(\rho E \vec{v}\right) = \nabla\cdot\left[\vec{W} - \vec{q} - \sum_{i=1}^{N}\left(\hat{\rho}_i \hat{E}_i \vec{\hat{W}}_i\right)\right] + \rho \vec{g}\cdot\vec{v} + \sum_{i=1}^{N} \hat{\rho}_i \vec{\hat{g}}_i \cdot \vec{W}_i$$

$$\vec{W} = \sum_{i=1}^{N} \vec{\hat{W}}_i, \quad \vec{q} = \sum_{i=1}^{N} \vec{\hat{q}}_i$$

We can also write the conservation of energy for phase j as:

$$\hat{\rho}_i \frac{D_i \hat{E}_i}{Dt} = \nabla\cdot\left(\vec{\hat{W}}_i - \vec{\hat{q}}_i\right) + \hat{\rho}_i \vec{\hat{g}}_i \cdot \vec{\hat{v}}_i + \sum_{\substack{j=1\\ j\neq i}}^{N}\left(E_{ji} - J_{ji}\hat{E}_i\right)$$

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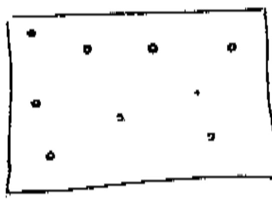
Dispersed particles in a fluid continuous phase

Small spherical particles in a fluid with volume fraction much smaller than one $\alpha << 1$.



Ω

$r_p << r_\Omega$

$N_\Omega >> 1$

The particles are small compared to the averaging volume and there are many of them.

Continuity

$$\frac{\partial (1-\alpha)\rho_f}{\partial t} + \nabla\cdot\left[(1-\alpha)\rho_f \vec{v}_f\right] = 0$$

$$\frac{\partial(\alpha\rho_s)}{\partial t} + \nabla\cdot\left[\alpha\rho_s\vec{v}_s\right] = 0$$

no phase change

Conservation of Momentum

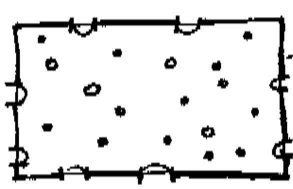
$$\frac{\partial}{\partial t}\left[(1-\alpha)\rho_f\vec{v}_f\right] + \nabla\cdot\left[(1-\alpha)\rho_f\vec{v}_f\vec{v}_f\right] = -\nabla\left[(1-\alpha)P_f\right] + \nabla\cdot\left[(1-\alpha)\bar{\bar{\tau}}'_f\right] -$$

$$-\nabla\cdot\left[(1-\alpha)\bar{\bar{\Pi}}_f\right] + (1-\alpha)\rho_f\vec{g} - \underbrace{\frac{1}{\Omega}\int_{S_i\cap\Omega} P_f'\,\vec{n}\,dS_i} + \underbrace{\frac{1}{\Omega}\int_{S_i\cap\Omega}\bar{\bar{\tau}}'_f\cdot\vec{n}\,dS_i}$$

The tildes mean relative press and velocity

$$-\nabla\left[(1-\alpha)P_f\right] = -\nabla\cdot\left(\frac{1}{\Omega}\int_{V_f} p\,\bar{\bar{I}}\,dV\right) = -\frac{1}{\Omega}\int_{V_f}\nabla\cdot\left(p\,\bar{\bar{I}}\right)dV =$$

$$= -\frac{1}{\Omega}\int_{\partial V_f} p\,\bar{\bar{I}}\cdot\vec{n}\,dS = -\frac{1}{\Omega}\int_{\partial\Omega\cap V_f} p\,\bar{\bar{I}}\cdot\vec{n}\,dS$$



$\partial\Omega\cap V_f$

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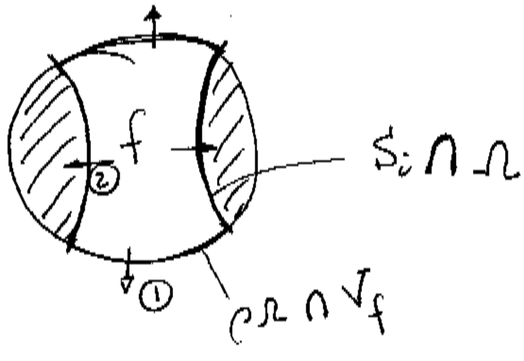
This term $-\frac{1}{\Omega}\int_{\partial\Omega\cap V_f} p\,\bar{\bar{I}}\cdot\vec{n}\,dS$ together with $-\frac{1}{\Omega}\int_{S_i\cap\Omega} p_f'\,\vec{n}\,dS$

represent the total effect of pressure on the fluid



$$\underbrace{-\frac{1}{\Omega}\int_{\partial\Omega\cap V_f} p\,\bar{\bar{I}}\cdot\vec{n}\,dS}_{1} - \frac{1}{\Omega}\int_{S_i\cap\Omega} p_f'\,\vec{n}\,dS \approx -\frac{1}{\Omega}\int_{\partial\Omega} p_f'\,\vec{n}\,dS = -\frac{1}{\Omega}\int_{\Omega}\nabla p_f'\,d$$

So we get:

$$(1-\alpha)\rho_f\frac{\partial\vec{v}_f}{\partial t} + (1-\alpha)\rho_f\,\vec{v}_f\cdot\nabla\vec{v}_f = -\nabla p_f + \nabla\cdot\left[(1-\alpha)\bar{\bar{\tau}}_f'\right] -$$

$$-\nabla\cdot\left[(1-\alpha)\bar{\bar{\pi}}_f\right] + (1-\alpha)\rho_f\vec{g} + \frac{1}{\Omega}\int_{S_i\cap\Omega}\bar{\bar{\tau}}_f'\cdot\vec{n}\,dS_i$$

We can also group together the viscous stresses:

$$\nabla\cdot\frac{1}{\Omega}\int_{V_f}\bar{\bar{\tau}}_f\cdot\vec{n}\,dV + \frac{1}{\Omega}\int_{S_i\cap\Omega}\bar{\bar{\tau}}_f'\cdot\vec{n}\,dS_i \approx \nabla\cdot\bar{\bar{\tau}}_m$$

$\bar{\bar{\tau}}_m$ is the equivalent viscous stress tensor:

$$\tau_m = \hat{\mu}\left[\nabla\vec{v}_m + (\nabla\vec{v}_m)^T\right] - \frac{2}{3}\hat{\mu}(\nabla\cdot\vec{v}_m)\,\bar{\bar{I}}$$

$V_m = (1-\alpha)\vec{v}_f + \alpha\vec{v}_s$ and $\hat{\mu} = \mu_f\left(1 + \alpha\frac{\mu_f + 5/2\,\mu_s}{\mu_f + \mu_s}\right)$

$(1+\frac{5}{2}\alpha)\mu_f$ for solids

for bubbles $(1+\alpha)\mu_f$